Variation of a nonlocal modified Einstein gravity action*

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Abstract

We consider nonlocal modified Einstein gravity without matter, where nonlocal term is of the form $P(R)\mathcal{F}(\Box)Q(R)$. Equations of motion are usually very complex. In this paper we present the derivation of the first and second variation of the action (1). The equations of motion are given. We give the expressions for perturbations in scalar, vector and tensor form.

1. Introduction

Although very successful, Einstein theory of gravity is not a final theory. There are many its modifications, which are motivated by quantum gravity, string theory, astrophysics and cosmology (for a review, see [2]). One of very promising directions of research is *nonlocal modified gravity* and its applications to cosmology (as a review, see [3] and [4]).

Under nonlocal modification of gravity we understand replacement of the scalar curvature R in the Einstein-Hilbert action by a suitable function $F(R, \Box, \Box^{-1}, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \dots)$, where $\Box = \nabla_\mu \nabla^\mu$ is d'Alembert operator and ∇_μ denotes the covariant derivative. As a review see [7, 6, 5] and references therein. Here, nonlocality means that Lagrangian contains an infinite number of space-time derivatives, i.e. derivatives up to an infinite order in the form of d'Alembert operator $\Box$ which is argument of an analytic function.

In the sequel we consider nonlocal modification of gravity where Einstein-Hilbertaction contains an additional nonlocal term of the form $P(R)\mathcal{F}(\Box)Q(R)$. In particular we consider a class of nonlocal gravity models without matter,

* Work partially supported by the Serbian Ministry of Education, Science and Technological Development, contract No. 174012.

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given by the following action

$$S = \frac{1}{16\pi G}\int_M \left(R - 2\Lambda + P(R)\mathcal{F}(\Box)Q(R)\right)\,\sqrt{-g}\,\mathrm{d}^4x, \tag{1}$$

where M is pseudo-Riemann manifold of signature $(1,3)$ with metric $(g_{\mu\nu})$, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$, P and Q are differentiable functions of the scalar curvature R and Λ is cosmological constant. The corresponding Einstein equations of motion are complex. In this paper we will present their derivation. In order to obtain equations of motion for $g_{\mu\nu}$ we have to find the variation of the action (1) with respect to metric $g^{\mu\nu}$. In addition we also find the second variation of the action (1) and consider some cosmic perturbations.

2. Variation of curvature tensors

Let us start with a technical lemma:

Lemma 1. *The following relations hold*

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu}, \tag{2}$$

$$\delta\sqrt{-g} = -\frac{1}{2}g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}, \tag{3}$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{2}\left(g_{\nu\alpha}\nabla_\mu\delta g^{\lambda\alpha} + g_{\mu\alpha}\nabla_\nu\delta g^{\lambda\alpha} - g_{\mu\alpha}g_{\nu\beta}\nabla^\lambda\delta g^{\alpha\beta}\right), \tag{4}$$

where g is the determinant of the metric tensor.

Lemma 2. *The variation of Riemman tensor, Ricci tensor and scalar curvature satisfy the following relations*

$$\delta R^{\alpha}_{\mu\beta\nu} = \nabla_\beta\delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\alpha}_{\mu\beta}, \tag{5}$$

$$\delta R_{\mu\nu} = \nabla_\lambda\delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\lambda}_{\mu\lambda}, \tag{6}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} - K_{\mu\nu}\delta g^{\mu\nu}, \tag{7}$$

$$\delta\nabla_\mu\nabla_\nu\psi = \nabla_\mu\nabla_\nu\delta\psi - \nabla_\lambda\psi\delta\Gamma^{\lambda}_{\mu\nu}, \tag{8}$$

where $K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$.

Lemma 3. *Every scalar function $P(R)$ satisfies*

$$\int_M P K_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \int_M K_{\mu\nu}P\;\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \tag{9}$$

Proof. At the beginning we prove $\int_M Pg_{\mu\nu}(\Box\delta g^{\mu\nu})\sqrt{-g}\,\mathrm{d}^4x = \int_M g_{\mu\nu}(\Box P)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x$ by application of Stokes' theorem:

$$\begin{aligned}\int_M Pg_{\mu\nu}\Box\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x &= -\int_M \nabla_\alpha(Pg_{\mu\nu})\nabla^\alpha\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\ &= \int_M g_{\mu\nu}\Box P\;\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x.\end{aligned} \tag{10}$$

Let $Z^\mu = P\nabla_\nu\delta g^{\mu\nu} - \nabla_\nu P\delta g^{\mu\nu}$. Integration over M yields $\int_M \nabla_\mu Z^\mu\sqrt{-g}\,\mathrm{d}^4x = \int_{\partial M} Z^\mu n_\mu \mathrm{d}\partial M$, where n_μ is the unit normal to a hypersurface ∂M. Since the restriction $Z^\mu|_{\partial M}$ vanish, we obtain

$$\int_M P\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \int_M \nabla_\mu\nabla_\nu P\;\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \qquad \Box$$

Lemma 4. *Let $P(R)$ and $Q(R)$ be scalar functions. Then for all $n\in\mathbb{N}$*

$$\begin{aligned}\int_M P\delta\Box^n Q\sqrt{-g}\,\mathrm{d}^4x = \frac{1}{2}\sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l P, \Box^{n-1-l}Q)\delta g^{\mu\nu}\;\sqrt{-g}\,\mathrm{d}^4x \\ + \int_M \Box^n P\;\delta Q\sqrt{-g}\,\mathrm{d}^4x.\end{aligned} \tag{11}$$

Proof. The definition of the $\Box$ operator implies

$$\begin{aligned}I &= \int_M P\delta\Box^n Q\sqrt{-g}\,\mathrm{d}^4x = \int_M P\delta(g^{\mu\nu}\nabla_\mu\nabla_\nu\Box^{n-1}Q)\sqrt{-g}\,\mathrm{d}^4x \\ &= \int_M P\left(\nabla_\mu\nabla_\nu\;\Box^{n-1}Q\delta g^{\mu\nu} + g^{\mu\nu}\delta\nabla_\mu\nabla_\nu\Box^{n-1}Q\right)\sqrt{-g}\,\mathrm{d}^4x \\ &= \int_M P\Big(\nabla_\mu\nabla_\nu\;\Box^{n-1}Q\delta g^{\mu\nu} + \Box\delta\Box^{n-1}Q - \nabla_\lambda\Box^{n-1}Qg^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\Big)\sqrt{-g}\,\mathrm{d}^4x.\end{aligned} \tag{12}$$

Moreover, from the Lemma 3 and Stokes' theorem we get

$$\begin{aligned}I &= \int_M P\Big(\nabla_\mu\nabla_\nu\Box^{n-1}Q\delta g^{\mu\nu} + \Box\delta\Box^{n-1}Q \\ &+ \frac{1}{2}\nabla_\lambda\Box^{n-1}Q(2\nabla_\mu\delta g^{\lambda\mu} - g_{\mu\nu}\nabla^\lambda\delta g^{\mu\nu})\Big)\sqrt{-g}\,\mathrm{d}^4x \\ &= \int_M P\;\Box\delta\Box^{n-1}Q\sqrt{-g}\,\mathrm{d}^4x - \int_M \nabla_\mu P\nabla_\nu\Box^{n-1}Q\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\ &- \frac{1}{2}\int_M g_{\mu\nu}(\nabla^\lambda P\nabla_\lambda\Box^{n-1}Q + P\Box\Box^{n-1}Q)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\ &= \int_M P\;\Box\delta\Box^{n-1}Q\sqrt{-g}\,\mathrm{d}^4x + \frac{1}{2}\int_M S_{\mu\nu}(P,\Box^{n-1}Q)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x.\end{aligned} \tag{13}$$

Partial integration in the first term of the pervious formula completes the proof.

$\square$

Theorem 1. *Let P and Q be scalar functions of scalar curvature, then*

$$\int_M P\delta(\sqrt{-g})\,\mathrm{d}^4x = -\frac{1}{2}\int_M g_{\mu\nu}P\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x, \tag{14}$$

$$\int_M P\delta R\sqrt{-g}\,\mathrm{d}^4x = \int_M (R_{\mu\nu}P - K_{\mu\nu}P)\,\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x, \tag{15}$$

$$\begin{aligned}\int_M P\delta(\mathcal{F}(\Box)Q)\sqrt{-g}\,\mathrm{d}^4x &= \int_M (R_{\mu\nu} - K_{\mu\nu})\left(Q'\mathcal{F}(\Box)P\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\ &+ \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l P, \Box^{n-1-l}Q)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x,\end{aligned} \tag{16}$$

where $S_{\mu\nu}(A,B) = g_{\mu\nu}\nabla^\alpha A\nabla_\alpha B + g_{\mu\nu}A\Box B - 2\nabla_\mu A\nabla_\nu B.$

Proof. Equation (14) is a consequence of (3).
From Lemma 2 and Lemma 3 we get

$$\begin{aligned}\int_M P\delta R\sqrt{-g}\,\mathrm{d}^4x &= \int_M (R_{\mu\nu}P\delta g^{\mu\nu} - PK_{\mu\nu}\delta g^{\mu\nu})\sqrt{-g}\,\mathrm{d}^4x \\ &= \int_M (R_{\mu\nu}P - K_{\mu\nu}P)\,\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x.\end{aligned} \tag{17}$$

To prove (16) let us introduce the following notation $J_n = \int_M P\delta(\Box^n Q)\sqrt{-g}\,\mathrm{d}^4x$. Then $\int_M P\delta(\mathcal{F}(\Box)Q)\sqrt{-g}\,\mathrm{d}^4x = \sum_{n=0}^{\infty} f_n J_n$. The integral J_0 is calculated by applying (15). For $n > 0$ integral J_n is calculated by applying Lemma 4 and (15).

$$\begin{aligned}J_n &= \int_M \left(R_{\mu\nu}Q'\Box^n P - K_{\mu\nu}(Q'\Box^n P)\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\ &+ \frac{1}{2}\sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l P, \Box^{n-1-l}Q)\delta g^{\mu\nu}\ \sqrt{-g}\,\mathrm{d}^4x.\end{aligned} \tag{18}$$

Summation over n yields equation (16). $\square$

3. Equations of motion

Let us consider the action (1). In order to calculate δS we introduce the following auxiliary actions

$$S_0 = \int_M (R - 2\Lambda)\,\sqrt{-g}\,\mathrm{d}^4x, \tag{19}$$

$$S_1 = \int_M P(R)\mathcal{F}(\Box)Q(R)\,\sqrt{-g}\,\mathrm{d}^4x. \tag{20}$$

Action S_0 is Einstein-Hilbert action and its variation is

$$\delta S_0 = \int_M (G_{\mu\nu} + \Lambda g_{\mu\nu})\,\delta g^{\mu\nu}\,\sqrt{-g}\,\mathrm{d}^4x. \tag{21}$$

Lemma 5. *Variation of the action S_1 is*

$$\begin{aligned}\delta S_1 = &-\frac{1}{2}\int_M g_{\mu\nu}P(R)\mathcal{F}(\Box)Q(R)\delta g^{\mu\nu}\,\sqrt{-g}\,\mathrm{d}^4x \\ &+\int_M (R_{\mu\nu}W - K_{\mu\nu}W)\,\delta g^{\mu\nu}\,\sqrt{-g}\,\mathrm{d}^4x \\ &+\frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l P(R), \Box^{n-1-l}Q(R))\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x, \end{aligned} \tag{22}$$

where $W = P'(R)\mathcal{F}(\Box)Q(R) + Q'(R)\mathcal{F}(\Box)P(R)$.

Theorem 2. *Variation of the action* (1) *is equal to zero iff*

$$\hat{G}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}P(R)\mathcal{F}(\Box)Q(R) + (R_{\mu\nu}W - K_{\mu\nu}W) + \frac{1}{2}\Omega_{\mu\nu} = 0, \tag{23}$$

where

$$W = P'(R)\mathcal{F}(\Box)Q(R) + Q'(R)\mathcal{F}(\Box)P(R), \tag{24}$$

$$\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} S_{\mu\nu}\left(\Box^l P(R), \Box^{n-1-l}Q(R)\right). \tag{25}$$

Proof. The proof of the Theorem is evident from the Lemma 5 and Theorem 1. □

4. Second variation of the action

In the following section we set $h_{\mu\nu} = \delta g_{\mu\nu}$. From Lemma 2 we see that $h^{\mu\nu} = -\delta g^{\mu\nu}$. Also let $h = g^{\mu\nu}h_{\mu\nu}$ be the trace of $h_{\mu\nu}$.

Operator $\delta\Box$ is defined by $(\delta\Box)V = \delta(\Box V) - \Box\delta V$. Then we can prove the following Lemma

Lemma 6. *Let U, V be scalar functions. Then*

$$(\delta\Box)V = -h^{\mu\nu}\nabla_\mu\nabla_\nu V - \nabla^\mu h_\mu^\lambda \nabla_\lambda V + \frac{1}{2}\nabla^\lambda h \nabla_\lambda V, \tag{26}$$

$$\int_M U(\delta\Box)V\sqrt{-g}\,\mathrm{d}^4x = \frac{1}{2}\int_M S_{\mu\nu}(U,V)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \tag{27}$$

In the next lemma we find the variation of $\mathcal{F}(\Box)$.

Lemma 7. *Let U, V be scalar functions. Then,*

$$\int_M U\delta(\mathcal{F}(\Box))V\sqrt{-g}\,\mathrm{d}^4x = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l U, \Box^{n-1-l}V)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \tag{28}$$

Proof. Note that $\delta\Box^n = \sum_{l=0}^{n-1}\Box^l(\delta\Box)\Box^{n-1-l}$ for $n > 0$ and $\delta\Box^0 = \delta\mathrm{Id} = 0$. Therefore sumation over n and integration yields

$$\int_M U\delta(\mathcal{F}(\Box))V\sqrt{-g}\,\mathrm{d}^4x = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M U\Box^l(\delta\Box)\Box^{n-1-l}V\sqrt{-g}\,\mathrm{d}^4x \tag{29}$$

$$= \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M \Box^l U(\delta\Box)\Box^{n-1-l}V\sqrt{-g}\,\mathrm{d}^4x \tag{30}$$

$$= \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l U, \Box^{n-1-l}V)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \tag{31}$$

□

Lemma 8. *Let U be scalar function. Then,*

$$\int_M U\delta W\sqrt{-g}\,\mathrm{d}^4x = \int_M (R_{\mu\nu}Y - K_{\mu\nu}Y + \frac{1}{2}\Psi_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x, \tag{32}$$

$$Y = U(P''\mathcal{F}(\Box)Q + Q''\mathcal{F}(\Box)P) + (P'\mathcal{F}(\Box)(Q'U) + Q'\mathcal{F}(\Box)(P'U)), \tag{33}$$

$$\Psi_{\mu\nu} = \sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1}\Big(S_{\mu\nu}(\Box^l(P'U), \Box^{n-1-l}Q) + S_{\mu\nu}(\Box^l(Q'U), \Box^{n-1-l}P)\Big) \tag{34}$$

Lemma 9. *Let A, B be scalar functions. Then,*

$$\int_M S_{\mu\nu}(\delta A, B)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \int \sigma_1(B)\delta A\sqrt{-g}\,\mathrm{d}^4x, \tag{35}$$

$$\int_M S_{\mu\nu}(A, \delta B)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \int \sigma_2(A)\delta B\sqrt{-g}\,\mathrm{d}^4x, \tag{36}$$

where

$$\sigma_1(B) = \nabla^\lambda h\nabla_\lambda B - 2\nabla_\mu h^{\mu\nu}\nabla_\nu B - 2h^{\mu\nu}\nabla_\mu\nabla_\nu B, \tag{37}$$

$$\sigma_2(A) = -\nabla^\lambda h\nabla_\lambda A - A\Box h - 2\nabla_\nu h^{\mu\nu}\nabla_\mu A - 2h^{\mu\nu}\nabla_\mu\nabla_\nu A. \tag{38}$$

Proof. To prove the first equation recall the definition of $S_{\mu\nu}(A, B)$

$$\int_M S_{\mu\nu}(\delta A, B)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \tag{39}$$

$$= \int_M \left(\nabla^\alpha(h\nabla_\alpha B) - h\Box B - 2\nabla_\mu(h^{\mu\nu}\nabla_\nu B)\right)\delta A\sqrt{-g}\,\mathrm{d}^4x \tag{40}$$

$$= \int_M \sigma_1(B)\delta A\sqrt{-g}\,\mathrm{d}^4x. \tag{41}$$

The proof of the second equation is similar. □

Lemma 10. *Let $\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} S_{\mu\nu}\left(\Box^l P(R), \Box^{n-1-l}Q(R)\right)$. Then,*

$$\begin{aligned}
&\int_M \delta\Omega_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \\
&= \int_M \sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1} \Big(h_{\mu\nu}\nabla^\lambda\Box^l P\nabla_\lambda\Box^{n-1-l}Q + h\nabla_\mu\Box^l P\nabla_\nu\Box^{n-1-l}Q \\
&+ h_{\mu\nu}\Box^l P\Box^{n-l}Q - \frac{1}{2}S_{\mu\nu}(h\Box^l P, \Box^{n-1-l}Q) \\
&+ (R_{\mu\nu} - K_{\mu\nu})(P'\Box^l(\sigma_1(\Box^{n-1-l}Q)) + Q'\Box^l(\sigma_2(\Box^{n-1-l}P))) \\
&+ \frac{1}{2}\int_M \sum_{n=1}^{+\infty} f_n \sum_{l=1}^{n-1}\sum_{m=0}^{l-1} \Big(S_{\mu\nu}(\Box^m(\sigma_1(\Box^{n-1-l}Q)), \Box^{l-m-1}P) \\
&+ S_{\mu\nu}(\Box^m(\sigma_2(\Box^{n-1-l}P)), \Box^{l-m-1}Q)\Big)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x
\end{aligned} \tag{42}$$

Proof. Note that

$$\delta\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \delta S_{\mu\nu}\left(\Box^l P(R), \Box^{n-1-l}Q(R)\right). \tag{43}$$

Moreover,

$$\int_M \delta S_{\mu\nu}(A,B)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \int_M \left(S_{\mu\nu}(\delta A,B)+S_{\mu\nu}(A,\delta B)\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x$$
$$+\int_M \big(h_{\mu\nu}\nabla^\lambda A\nabla_\lambda B + h\nabla_\mu A\nabla_\nu B + h_{\mu\nu}A\Box B - \frac{1}{2}S_{\mu\nu}(hA,B)\big)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x.$$

Using this formula for each term in $\delta\Omega_{\mu\nu}$ yields the result of the Lemma. □

Theorem 3. *The second variation of the action* (1) *is given by*

$$\delta^2 S = \frac{1}{16\pi G}\int_M \left(U_{\mu\nu} + R_{\mu\nu}X - K_{\mu\nu}X + \frac{1}{2}\chi_{\mu\nu} + \frac{1}{4}\Theta_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x, \tag{44}$$

where

$$\begin{aligned} U_{\mu\nu} &= -\frac{1}{2}h_{\mu\nu}(R-2\Lambda+P\mathcal{F}(\Box)Q) + \delta R_{\mu\nu}(W+1) + \delta\Gamma^\lambda_{\mu\nu}\nabla_\lambda W \\ &\quad + h_{\mu\nu}\Box W - \frac{1}{2}S_{\mu\nu}(h,W), \end{aligned} \tag{45}$$

$$\begin{aligned} X &= \frac{1}{2}(h + P'h\mathcal{F}(\Box)Q + Q'\mathcal{F}(\Box)(Ph)) + \Big(\delta R(P''\mathcal{F}(\Box)Q + Q''\mathcal{F}(\Box)P) \\ &\quad + (P'\mathcal{F}(\Box)(Q'\delta R) + Q'\mathcal{F}(\Box)(P'\delta R))\Big), \end{aligned} \tag{46}$$

$$\begin{aligned} \chi_{\mu\nu} &= \frac{1}{2}\sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1} S_{\mu\nu}(\Box^l(Ph),\Box^{n-l-1}Q) \\ &\quad - \sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1}\Big(S_{\mu\nu}(\Box^l(P'M),\Box^{n-1-l}Q) + S_{\mu\nu}(\Box^l(Q'M),\Box^{n-1-l}P)\Big) \\ &\quad + \sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1}\Big(h_{\mu\nu}\nabla^\lambda\Box^l P\nabla_\lambda\Box^{n-1-l}Q + h\nabla_\mu\Box^l P\nabla_\nu\Box^{n-1-l}Q \\ &\quad + h_{\mu\nu}\Box^l P\Box^{n-l}Q - \frac{1}{2}S_{\mu\nu}(h\Box^l P,\Box^{n-1-l}Q) \\ &\quad + (R_{\mu\nu}-K_{\mu\nu})(P'\Box^l(\sigma_1(\Box^{n-1-l}Q)) + Q'\Box^l(\sigma_2(\Box^{n-1-l}P)))\Big), \end{aligned} \tag{47}$$

and

$$\Theta_{\mu\nu} = \sum_{n=1}^{+\infty} f_n \sum_{l=1}^{n-1} \sum_{m=0}^{l-1} \Big(S_{\mu\nu}(\Box^m(\sigma_1(\Box^{n-1-l}Q)), \Box^{l-m-1}P) + S_{\mu\nu}(\Box^m(\sigma_2(\Box^{n-1-l}P)), \Box^{l-m-1}Q) \Big), \tag{48}$$

$$\sigma_1(B) = \nabla^\lambda h \nabla_\lambda B - 2\nabla_\mu h^{\mu\nu}\nabla_\nu B - 2h^{\mu\nu}\nabla_\mu\nabla_\nu B, \tag{49}$$

$$\sigma_2(A) = -\nabla^\lambda h \nabla_\lambda A - A\Box h - 2\nabla_\nu h^{\mu\nu}\nabla_\mu A - 2h^{\mu\nu}\nabla_\mu\nabla_\nu A. \tag{50}$$

Proof. In the pervious section we calculated the first variation of the action (1)

$$\delta S = \frac{1}{16\pi G}\int_M \hat{G}_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x. \tag{51}$$

Moreover the second variation $\delta^2 S$ is

$$\delta^2 S = \frac{1}{16\pi G}\int_M \left(\delta\hat{G}_{\mu\nu}\delta g^{\mu\nu} + \hat{G}_{\mu\nu}\delta^2 g^{\mu\nu} - \frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}\hat{G}_{\mu\nu}\delta g^{\mu\nu}\right)\sqrt{-g}\ \mathrm{d}^4x. \tag{52}$$

At the beginning note that

$$\int_M \delta\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x = \int_M \left(\delta R_{\mu\nu} - \frac{1}{2}(R-2\Lambda)h_{\mu\nu} + \frac{1}{2}R_{\mu\nu}h - \frac{1}{2}K_{\mu\nu}h\right)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x. \tag{53}$$

The next term is calculated by using Lemma 7

$$\int_M \delta\left(g_{\mu\nu}P\mathcal{F}(\Box)Q\right)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x = \int_M h_{\mu\nu}P\mathcal{F}(\Box)Q\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x - \frac{1}{2}\sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1}\int S_{\mu\nu}(\Box^l(Ph), \Box^{n-l-1}Q)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x - \int_M (R_{\mu\nu} - K_{\mu\nu})\left(P'h\mathcal{F}(\Box)Q + Q'\mathcal{F}(\Box)(Ph)\right)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x. \tag{54}$$

The third term is $\int_M \delta\left(R_{\mu\nu}W\right)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x$ and it is equal to

$$\int_M \delta\left(R_{\mu\nu}W\right)\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x \tag{55}$$

$$= -\frac{1}{2}\int_M \left(\Box h_{\mu\nu} + \nabla_\mu\nabla_\nu h - 2\nabla_\lambda\nabla_\mu h^\lambda_\nu\right) W\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x + \int_M R_{\mu\nu}\delta W\delta g^{\mu\nu}\sqrt{-g}\ \mathrm{d}^4x \tag{56}$$

The last integral of the above formula is obtained by Lemma 8. Similarly, we obtain

$$\int_M \delta\left(K_{\mu\nu}W\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x = \int_M \delta W K_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x$$
$$-\int\left(\delta\Gamma^{\lambda}_{\mu\nu}\nabla_\lambda W + h_{\mu\nu}\Box W - \frac{1}{2}S_{\mu\nu}(h,W)\right)\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \tag{57}$$

At the end the last term $\int_M \delta\Omega_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x$ is calculated in Lemma 10.

□

5. Perturbations

5.1. Background

In this section we look for FRW metric with $k = 0$ which can be written as

$$ds^2 = -dt^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{58}$$

Some relevant background quantities are

$$R = 12H^2 + 6\dot{H},\ \Gamma^0_{ij} = Hg_{ij},\ \Gamma^i_{j0} = H\delta^i_j,\ \Box = -\partial_t^2 - 3H\partial_t + \frac{\delta^{ij}\partial_i\partial_j}{a^2} \tag{59}$$

where the indexes i, j range as $1, 2, 3$. On the background all quantities are space homogeneous as the metric suggests.

For perturbations we employ the canonical ADM $(1+3)$ decomposition and introduce the conformal time τ such that

$$a\mathrm{d}\tau = dt$$

Then the general FRW metric (58) transforms

$$ds^2 = a(\tau)^2(-d\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{60}$$

5.2. Perturbations

The metric perturbations (see [1]) can be categorized in three types: scalar, vector and tensor perturbations. The component h_{00} is invariant under spatial rotations and translations and therefore

$$h_{00} = 2a(\tau)^2\phi. \tag{61}$$

The components h_{0i} are the sum of a spatial gradient of a function B and divergence free vector S_i.

$$h_{0i} = a(\tau)^2(\partial_i B + S_i). \tag{62}$$

Similarly, components h_{ij}, which transform as a tensor under 3− rotations are written as

$$h_{ij} = a(\tau)^2(2\psi\delta_{ij} + 2\partial^2_{ij}E + \partial_i F_j + \partial_j F_i + \varphi_{ij}), \tag{63}$$

where ψ and E are scalar functions, F_i is a vector with zero divergence and 3− tensor satisfies

$$\varphi^i_i = 0, \qquad \partial_i\varphi^i_j = 0. \tag{64}$$

Note that three are four scalar functions, two vectors with two independent components each and tensor s_{ij} has two independent components. Therefore, as expected we have total of ten functions.

The scalar perturbations are defined by scalar functions ϕ, ψ, B, E and perturbed metric around FRW b background is

$$ds^2 = a(\tau)^2\left[-(1-2\phi)d\tau^2 + \partial_i B \mathrm{d}\tau dx^i + ((1+2\psi)\delta_{ij} + 2\partial_i\partial_j E)dx^i dx^j\right], \tag{65}$$

The vector perturbations are defined by vectors S_i and F_i.

$$ds^2 = a(\tau)^2\left[-d\tau^2 + S_i B \mathrm{d}\tau dx^i + (\delta_{ij} + \partial_i F_j + \partial_j F_i)dx^i dx^j\right], \tag{66}$$

Tensor perturbations are defined by φ_{ij} and describe gravity waves, and have no analog in Newtonian theory.

$$ds^2 = a(\tau)^2\left[-d\tau^2 + (\delta_{ij} + \varphi_{ij})dx^i dx^j\right], \tag{67}$$

Each of the types of perturbations can be studied separately. In this form of perturbations we have

$$h_{\mu\nu} = a(\tau)^2\begin{pmatrix} 2\phi & \partial_i B + S^i \\ \partial_i B + S^i & 2\psi\mathrm{Id} + 2\,\mathrm{Hess}\,E + \partial_j F_i + \partial_i F_j + \varphi_{ij}\end{pmatrix}, \tag{68}$$

$$h = -2\phi + 6\psi + 2\triangle E, \tag{69}$$

$$\begin{aligned}\delta R = \frac{2}{a^2}\big(&6\frac{a''}{a}\phi + \triangle(\phi - 2\psi) + 3\frac{a'}{a}\triangle(B + E') \\ &+ 3\frac{a'}{a}(\phi' + 3\psi') + \triangle(B' + E'') + 3\psi''\big).\end{aligned} \tag{70}$$

Moreover, let $A^\mu = \nabla_\nu h^{\mu\nu}$, then

$$A^0 = 6\frac{a'}{a^3}(\phi + \psi) + \frac{1}{a^2}\triangle B + 2\frac{a'}{a^3}\triangle E + \frac{2}{a^2}\phi', \tag{71}$$

$$A^i = 4\frac{a'}{a^3}(\partial_i B + S^i) + \frac{2}{a^2}(\partial_i\psi + \triangle\partial_i E) + \frac{1}{a^2}(\partial_i B' + S'^i + \triangle F^i). \tag{72}$$

Let $r_{\mu\nu} = \delta R_{\mu\nu}$, then

$$r_{00} = -\frac{1}{a}\Big(a'(\triangle(B+E') + 3(\phi' + \psi')) + a(\triangle(\phi + B' + E'') + 3\psi'')\Big), \tag{73}$$

$$r_{i0} = -\frac{1}{a^2}\Big((a'^2 + aa'')\partial_i B + 2aa'\partial_i\phi\Big) - 2\partial_i\psi', \tag{74}$$

$$\begin{aligned} r_{ii} = \frac{1}{a^2}\Big(&2(a'^2 + aa'')(\phi + \psi + \partial_{ii}^2 E) + aa'(\triangle(B+E') \\ &+ 2\partial_{ii}^2(B+E') + \phi' + 5\psi') + a^2(\partial_{ii}^2(\phi + B' + E'' - \psi) + \psi'' - \triangle\psi)\Big), \end{aligned} \tag{75}$$

$$r_{ij} = \frac{1}{a^2}\Big(2\partial_{ij}^2 E(a'^2 + aa'') + 2aa'\partial_{ij}^2(B+E') + a^2\partial_{ij}^2(\phi - \psi + B' + E'')\Big), \ i \neq j. \tag{76}$$

Out of 4 scalar modes only 2 are gauge invariant. The convenient gauge invariant variables (Bardeen potentials) are introduced as

$$\Phi = \phi - \frac{1}{a}(a(B-E'))', \qquad \Psi = \psi + \frac{a'}{a}(B-E'). \tag{77}$$

The prime denotes the differentiation with respect to the conformal time τ and the dot as before w.r.t. the cosmic time t.

The $(1+3)$ structure suggests to represent the perturbation quantities (which can depend on all 4 coordinates) as

$$f(\tau, \vec{x}) = f(\tau, k)Y(k, \vec{x}) \tag{78}$$

where $\vec{x} = (x, y, z)$ and $k = |\vec{k}|$ comes from the definition of the Y-functions as spatial Fourier modes

$$\delta^{ij}\partial_i\partial_j Y = -k^2 Y \tag{79}$$

Obviously

$$Y = Y_0 e^{\pm i\vec{k}\vec{x}} \tag{80}$$

The relevant expressions for the d'Alembertian operator are

$$\Box = -\frac{1}{a^2}\partial_\tau^2 - 2\frac{a'}{a^3}\partial_\tau - \frac{k^2}{a^2} = -\partial_t^2 - 3H\partial_t - \frac{k^2}{a^2} \tag{81}$$

Recall that considering the de Sitter background we do not need the perturbed d'Alembertian operator to the linear order in perturbation analysis. However, all the expression in this subsection are valid for a generic scale factor a.

6. Concluding Remarks

In this paper we have considered a nonlocal gravity model without matter given by the action in the form

$$S = \frac{1}{16\pi G} \int_M \left(R - 2\Lambda + P(R)\mathcal{F}(\Box)Q(R)\right) \sqrt{-g}\, \mathrm{d}^4x. \tag{82}$$

We have derived the equations of motion for this action. We also have presented the second variation of the previous action.

In many research papers there are equations of motion which are a special case of our equations. In the case $P(R) = Q(R) = R$ we obtain

$$S = \frac{1}{16\pi G} \int_M \left(R - 2\Lambda + R\mathcal{F}(\Box)R\right) \sqrt{-g}\, \mathrm{d}^4x.$$

This nonlocal model is elaborated is the series of papers [8, 9, 10, 11, 12, 13, 14, 15].

The action (82) for $P(R) = R^{-1}$ and $Q(R) = R$ was introduced in [16] as a new approach to nonlocal gravity. This model one can also find in [17].

The case $P(R) = R^p$ and $Q(R) = R^q$ we analyze in [19, 20].

For the case $P(R) = (R + R_0)^m$ and $Q(R) = (R + R_0)^m$ see [18].

In the last case we have $R = \text{const}$. Studies of this model can be found in [21, 22, 19].

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