Quasi-classical limit of the open Jordanian XXX spin chain*

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Abstract

We study the open deformed XXX spin chain. In particular we obtain the explicit expression of the Sklyanin monodromy matrix in terms of the entries of the local Lax operator of the Jordanian chain. These results are essential in the study of the so-called quasi-classical limit of the system.

1. Introduction

A particularly interesting feature of quantum groups is the so-called twist transformation [1]. It yields new quantum groups form already known ones. More precisely, a twist of a quantum group, or more generally, of a Hopf algebra $\mathcal{A}$ is a similarity transformation of the co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by an invertible twist element

$$\Delta(a) \mapsto \Delta_t(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \qquad \forall a \in \mathcal{A}. \tag{1}$$

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In order to guarantee the co-associativity property of the twisted co-product the element $\mathcal{F}$ must satisfy the so-called twist equation [1]

$$\mathcal{F}_{12}\left(\Delta\otimes\mathrm{id}\right)\left(\mathcal{F}\right)=\mathcal{F}_{23}\left(\mathrm{id}\otimes\Delta\right)\left(\mathcal{F}\right), \tag{2}$$

where

$$\left(\Delta\otimes\mathrm{id}\right)\mathcal{F}=\left(\Delta\otimes\mathrm{id}\right)\sum_j f_j^{(1)}\otimes f_j^{(2)}=\sum_j \Delta\left(f_j^{(1)}\right)\otimes f_j^{(2)}\in\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}. \tag{3}$$

Although the twist transformation generates an equivalence relation between quantum groups they produce different R-matrices. Namely, the transformation law of the co-product also determines how the corresponding universal R-matrix changes,

$$\mathcal{R}\mapsto\mathcal{R}^{(t)}=\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1},\quad \text{here}\quad \mathcal{F}_{21}=\sum_j f_j^{(2)}\otimes f_j^{(1)}. \tag{4}$$

This new R-matrix allows building and studying new integrable models [2, 3].

2. Deformed Yang R-matrix and the corresponding K-matrix

As our initial step, we briefly review the Jordanian twist element, as a particular solution of the twist equation. We consider $s\ell(2)$ generators S^α with $\alpha=+,-,3$, with the commutation relations

$$[S^3,S^\pm]=\pm S^\pm,\quad [S^+,S^-]=2S^3, \tag{5}$$

and Casimir operator

$$c_2=(S^3)^2+\frac{1}{2}(S^+S^-+S^-S^+)=(S^3)^2+S^3+S^-S^+=\vec{S}\cdot\vec{S}. \tag{6}$$

The universal enveloping algebra $U(s\ell(2))$ admits the Jordanian twist element [4, 5]

$$\mathcal{F}=\exp 2\left(S^3\otimes\sigma\right)\in U\left(s\ell(2)\right)\otimes U\left(s\ell(2)\right), \tag{7}$$

where

$$\sigma=\frac{1}{2}\log\left(1+2\theta S^+\right). \tag{8}$$

It straightforward to check that the Jordanian twist element satisfies the following equations [6]

$$\left(\Delta\otimes\mathrm{id}\right)\left(\mathcal{F}\right)=\mathcal{F}_{13}\mathcal{F}_{23},\qquad \left(\mathrm{id}\otimes\Delta_t\right)\left(\mathcal{F}\right)=\mathcal{F}_{12}\mathcal{F}_{13}. \tag{9}$$

In the equations above the co-product Δ is the usual co-product of the $U(s\ell(2))$ and Δ_t is the twisted co-product. Evidently the equations (9) imply the twist equation (2) [6]. Thus the the Jordanian twist element (7) satisfies the Drinfeld twist equation (2).

The XXX Heisenberg spin chain is related to the Yangian $\mathcal{Y}(s\ell(2))$ and the $SL(2)$-invariant Yang R-matrix [7]

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{10}$$

where λ is a spectral parameter, η is a quasi-classical parameter. We use $\mathbb{1}$ for the identity operator and $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The universal enveloping algebra of $s\ell(2)$ is a Hopf sub-algebra of the Yangian, $U(s\ell(2)) \subset \mathcal{Y}(s\ell(2))$. Notice that the matrix form of $\mathcal{F}$ in the spin-1/2 representation $\rho_{1/2}$ is $F_{12} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$,

$$F_{12} = \left(\rho_{1/2} \otimes \rho_{1/2}\right)(\mathcal{F}) = \mathbb{1} + 2\theta S^3 \otimes S^+ = \begin{pmatrix} 1 & \theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

in particular, in the spin-1/2 representation the generators S^α are given by the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

Therefore the transformation of the Yang R-matrix by the Jordanina twist element yields the R-matrix of the twisted Yangian $\mathcal{Y}_\theta(s\ell(2))$ [8, 9, 6]

$$R^J(\lambda) = F_{21} R_{12}(\lambda) F_{12}^{-1} = \begin{pmatrix} \lambda+\eta & -\lambda\theta & \lambda\theta & \lambda\theta^2 \\ 0 & \lambda & \eta & -\lambda\theta \\ 0 & \eta & \lambda & \lambda\theta \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{12}$$

where $F_{21} = \mathcal{P} F_{12} \mathcal{P}$. In what follows, we will use only twisted R-matrix (12) and in order to simplify the notation we will omit the symbol J in the superscript.

The R-matrix (12) satisfies the Yang-Baxter equation

$$R_{12}(\lambda-\mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda-\mu). \tag{13}$$

By setting $\theta = -\xi\eta$ we can guarantee the quasi-classical property

$$\frac{1}{\lambda+\eta} R(\lambda,\eta,\theta)|_{\theta=-\xi\eta} = \mathbb{1} + \eta r(\lambda) + O(\eta^2), \tag{14}$$

where $r(\lambda)$ is the classical r-matrix

$$r(\lambda) = \begin{pmatrix} 0 & \xi & -\xi & 0 \\ 0 & -\frac{1}{\lambda} & \frac{1}{\lambda} & \xi \\ 0 & \frac{1}{\lambda} & -\frac{1}{\lambda} & -\xi \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

which satisfies the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0 \tag{16}$$

and has the unitarity property $r_{21}(-\lambda) = -r_{12}(\lambda)$.

The R-matrix (12) has the so-called regularity property

$$R(0, \eta) = \eta \mathcal{P}, \tag{17}$$

and the unitarity property

$$R_{12}(\lambda) R_{21}(-\lambda) = g(\lambda) \mathbb{1}, \quad \text{with} \quad g(\lambda) = \eta^2 - \lambda^2. \tag{18}$$

The PT symmetry is broken

$$R_{21}(\lambda) \neq R_{12}^{t_1 t_2}(\lambda), \tag{19}$$

where the indices t_1 and t_2 denote the respective transpositions in the first and second space of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. The R-matrix does not have the crossing symmetry, but it satisfies the weaker condition

$$\left(\left(\left(R_{12}^{t_2}(\lambda) \right)^{-1} \right)^{t_2} \right)^{-1} = \frac{g(\lambda + \eta)}{g(\lambda + 2\eta)} M_2 R_{12}(\lambda + 2\eta) M_2^{-1}, \tag{20}$$

with

$$M = \begin{pmatrix} 1 & -2\theta \\ 0 & 1 \end{pmatrix}.$$

In [9] it was shown that the general solution to the reflection equation

$$R_{12}(\lambda - \mu) K_1^-(\lambda) R_{21}(\lambda + \mu) K_2^-(\mu) = K_2^-(\mu) R_{12}(\lambda + \mu) K_1^-(\lambda) R_{21}(\lambda - \mu) \tag{21}$$

is given by

$$K^-(\lambda) = \begin{pmatrix} \zeta + \lambda - \frac{\phi\theta}{\eta}\lambda^2 & \psi\lambda \\ \phi\lambda & \zeta - \lambda - \frac{\phi\theta}{\eta}\lambda^2 \end{pmatrix}. \tag{22}$$

Also, the dual reflection equation was derived and it was shown that its the general solution is given by [9]

$$K^+(\lambda) = K^-(-\lambda - \eta) M. \tag{23}$$

Final observation is that by setting $\theta = -\xi\eta$ we achieve that the matrix $K^-(\lambda)$ (22) does not depend on the parameter η i.e.,

$$\frac{\partial K^-(\lambda)}{\partial \eta} = 0. \tag{24}$$

This is an important step in the so-called quasi-classical limit of the corresponding chain [10, 11, 12, 13].

Using the results obtained above, in the next section, we will study the open deformed XXX spin chain following Sklyanin's approach [14] as we have successfully done in the case of XXX Heisenberg spin chain [10] and XXZ Heisenberg spin chain [12].

3. Jordanian deformation of the XXX spin chain

The Hilbert space of the system is

$$\mathcal{H} = \mathop{\otimes}_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}, \tag{25}$$

we study the deformed inhomogeneous spin chain with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$, corresponding inhomogeneous parameter α_m and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{26}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

We introduce the Lax operator

$$\begin{aligned}\mathbb{L}_{0m}(\lambda) = &\begin{pmatrix} e^{-\sigma_m} & 2\theta S_m^3 e^{\sigma_m} \\ 0 & e^{\sigma_m} \end{pmatrix} \\ &+ \frac{\eta}{\lambda} \begin{pmatrix} S_m^3 \left(\mathbb{1}_m + 2\theta S_m^+\right) e^{-\sigma_m} & \left(S_m^- - 2\theta (S_m^3)^2\right) e^{\sigma_m} \\ S_m^+ e^{-\sigma_m} & -S_m^3 e^{\sigma_m} \end{pmatrix}.\end{aligned} \tag{27}$$

In the case when the quantum space is a spin $\frac{1}{2}$ representation, the Lax operator is equal to the R-matrix, $\mathbb{L}_{0m}(\lambda) = \frac{1}{\lambda} R_{0m}(\lambda - \eta/2)$.

Due to the commutation relations (5), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda-\mu)\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0'm}(\mu-\alpha_m) = \mathbb{L}_{0'm}(\mu-\alpha_m)\mathbb{L}_{0m}(\lambda-\alpha_m)R_{00'}(\lambda-\mu). \tag{28}$$

The so-called monodromy matrix

$$T_0(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N) \cdots \mathbb{L}_{01}(\lambda - \alpha_1) \tag{29}$$

is used to describe the system. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{30}$$

From RLL-relations (28) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu). \tag{31}$$

The RTT-relations define the commutation relations for the entries of the monodromy matrix.

Also, we define the Lax operator

$$\begin{aligned}\widetilde{\mathbb{L}}_{0m}(\lambda) = &\begin{pmatrix} e^{\sigma_m} & -2\theta e^{\sigma_m} S_m^3 \\ 0 & e^{-\sigma_m} \end{pmatrix} \\ &+ \frac{\eta}{\lambda}\begin{pmatrix} e^{\sigma_m} S_m^3 & e^{\sigma_m}\left(S_m^- - 2\theta (S_m^3)^2\right) \\ e^{-\sigma_m} S_m^+ & -e^{-\sigma_m}\left(\mathbb{1}_m + 2\theta S_m^+\right) S_m^3 \end{pmatrix}.\end{aligned} \tag{32}$$

It obeys the following important identity

$$\mathbb{L}_{0m}(\lambda)\widetilde{\mathbb{L}}_{0m}(\eta-\lambda) = \Big(1 + \eta^2\,\frac{s_m(s_m+1)}{\lambda(\eta-\lambda)}\Big)\mathbb{1}_0\,, \tag{33}$$

where s_m is the value of spin in the space V_m. Thus the monodromy matrix

$$\widetilde{T}_0(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix} = \widetilde{\mathbb{L}}_{01}(\lambda+\alpha_1+\eta)\cdots\widetilde{\mathbb{L}}_{0N}(\lambda+\alpha_N+\eta), \tag{34}$$

obeys the following relations

$$\widetilde{T}_{0'}(\mu)R_{00'}(\lambda+\mu)T_0(\lambda) = T_0(\lambda)R_{00'}(\lambda+\mu)\widetilde{T}_{0'}(\mu), \tag{35}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu)R_{00'}(\mu-\lambda) = R_{00'}(\mu-\lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda). \tag{36}$$

By construction it follows that the entries of the Sklyanin monodromy matrix

$$\mathcal{T}_0(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} = T_0(\lambda)K_0^-(\lambda)\widetilde{T}_0(\lambda), \tag{37}$$

obey the exchange relations of the so-called reflection equation algebra [14, 10, 12]

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu). \tag{38}$$

4. Conclusions

The formulae (37), together with (27) and (32), yields, along the lines previously used successfully in the cases of the XXX and XXZ Heisenberg spin chains [10, 12], the quasi-classical expansion of the Sklyanin monodromy of the deformed chain. We believe that these results will help complete the study of the open deformed Gaudin model which we have initiated in [6]. Notice that the open trigonometric Gaudin was reviewed in [15]. The algebraic Bethe ansatz for the periodic deformed Gaudin model was done in [16, 17]. It is very likely that the implementation of the algebraic Bethe ansatz for the open deformed Gaudin model would require specific set of generators of the corresponding generalized Gaudin algebra, as in the $s\ell(2)$ case [18]. These considerations will be reported elsewhere.

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