Representation of T-duality of type II pure spinor superstring in double space*

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Abstract

In this article we present a new way of representation of T-duality using double space. Double space is obtained by adding T-dual coordinates to the initial ones. T-dualization in double space is represented by permutation of appropriate directions from initial and T-dual space. We did this both for bosonic and fermionic T-dualization of type II superstring theory propagating in the constant background fields. Obtained results are in full correspondence with the results of standard Buscher procedure.

1. Introduction

T-duality as a fundamental characteristic of string dynamics [1, 2, 3, 4, 5], unexprerienced by point-like particle, makes that there is no difference in physical content between string theories compactified on a circle of radius R and circle of radius $1/R$. It is very important for understanding M-theory, because five consistent superstring theories are connected by web of T and S dualities.

Buscher T-dualization procedure [2] represents a mathematical environment for performing T-duality. In order to make T-dualization along some directions, they should be isometry ones. Effectively, this means that background fields do not depend on those coordinates [2, 3, 4, 5, 6, 7]. Further, we localize noticed symmetry in standard way introducing world-sheet covariant derivatives, $\partial_\pm x^\mu \to D_\pm x^\mu = \partial_\pm x^\mu + v^\mu_\pm$, where $v^\mu_\pm$ are gauge fields. In order to have the same number of degrees of freedom in T-dual theory as in the initial one, the new term with Lagrange multipliers is added to the action. Using gauge freedom we fix initial coordinates and get the gauge

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fixed action. Varying gauge fixed action with respect to the Lagrange multipliers one gets the initial action and varying with respect to the gauge fields one gets T-dual action.

This standard T-dualization procedure was used in the papers [8, 9, 10, 11, 12] in the context of closed string noncommutativity. There is a generalized Buscher procedure which deals with background fields depending on all coordinates. The new step in this procedure is introducing of gauge invariant coordinate for the directions on which background fields depend on. The generalized procedure was applied to the case of bosonic string moving in the weakly curved background [13, 14]. It leads directly to closed string noncommutativity [15].

Double space formalism is a framework in which we can represent T-dualization in a simple and elegant way. It is spanned by double coordinates $Z^M = (x^\mu, y_\mu)$ $(\mu = 0, 1, 2, \dots, D-1)$, where x^μ and y_μ are the coordinates of the D-dimensional initial and T-dual space-time, respectively. It was the subject of the articles about twenty years ago [16, 17, 18, 19, 20], but interest for it occured again [21, 22, 23, 24, 25]. In all these papers T-duality is represented as $O(d, d)$ symmetry transformations.

In Refs.[26, 27] we doubled all bosonic coordinates and obtain the theory which contains the initial and all corresponding T-dual theories. In such theory partial T-dualization (T-dualization along some of the initial directions x^a) is represented as permutation of the corresponding coordinate subsets, x^a and y_a, which is a generalization of ideas given in [16].

When one says T-duality, one means bosonic T-duality. But since recently we can also speak about fermionic T-duality. Analyzing the gluon scattering amplitudes in $N = 4$ super Yang-Mills theory fermionic T-duality was discovered [28, 29]. Mathematically, fermionic T-duality is realized within Buscher procedure, except that dualization is performed along fermionic directions, θ^α and $\bar{\theta}^\alpha$.

In the present paper we are going to extend approach of double space to the type II theories both in the case of bosonic and fermionic T-duality [28, 30, 31]. We will show that double space method gives the same results as in the case of applying of standard Buscher procedure.

Here we start applying the approach of Refs.[26, 27] in the case of partial bosonic T-dualization of the type II superstring theory [1] and then in the case of fermionic T-duality. Generally, we use type II pure spinor action from [32]. This action is given in the form of an expansion in powers of fermionic coordinates θ^α and $\bar{\theta}^\alpha$. In both cases we simplify the action using different assumptions explained in the paper. As a result in both cases we obtain same action, which describes ghost free type II superstring theory in pure spinor formulation [33, 34, 35] in the approximation of constant background fields and up to the quadratic terms.

Bosonic T-dual transformation laws can be rewritten via double space coordinates Z^M. In order to achieve that we introduce the generalized metric $\mathcal{H}_{MN}$ and the generalized current $J_{\pm M}$. The permutation matrix $(\mathcal{T}^a)^M{}_N$ makes permutation of x^a and y_a, where index a marks the directions along which we make partial bosonic T-dualization. The T-dual coordinate is defined as ${}_aZ^M = (\mathcal{T}^a)^M{}_N Z^N$ and it has to obey the T-dual transformation law of the same form as initial coordinates, Z^M. This demand produces the expressions for T-dual generalized metric, ${}_a\mathcal{H}_{MN} = (\mathcal{T}^a \mathcal{H} \mathcal{T}^a)_{MN}$, and T-dual current, ${}_aJ_{\pm M} = (\mathcal{T}^a J_\pm)_M$. From transformation of the generalized metric we obtain T-dual NS-NS background fields and from transformation of the current we obtain T-dual NS-R fields. The transformation law for R-R field strength we get imposing additional assumtions because it is coupled by fermionic degrees of freedom along which we do not dualize.

Further we apply the method in the case of fermionic T-dualization. We are going to double fermionic sector of type II theories adding to the coordinates θ^α and $\bar\theta^\alpha$ their fermionic T-duals, ϑ_α and $\bar\vartheta_\alpha$, where $\alpha = 1, 2, \ldots, 16$. Rewriting T-dual transformation laws in terms of the double coordinates, $\Theta^A = (\theta^\alpha, \vartheta_\alpha)$ and $\bar\Theta^A = (\bar\theta^\alpha, \bar\vartheta_\alpha)$, we define the "fermionic generalized metric" $\mathcal{F}_{AB}$ and the generalized currents $\bar{\mathcal{J}}_{+A}$ and $\mathcal{J}_{-A}$. The permutation matrix $\mathcal{T}^A{}_B$ exchanges $\bar\theta^\alpha$ and θ^α with their T-dual partners, $\bar\vartheta_\alpha$ and ϑ_α, respectively. From the requirement that fermionic T-dual coordinates, ${}^\star\Theta^A = \mathcal{T}^A{}_B \Theta^B$ and ${}^\star\bar\Theta^A = \mathcal{T}^A{}_B \bar\Theta^B$, have the same T-dual transformation law as initial ones, Θ^A and $\bar\Theta^A$, we obtain the expressions for fermionic T-dual generalized metric, ${}^\star\mathcal{F}_{AB} = (\mathcal{T}\mathcal{F}\mathcal{T})_{AB}$, and T-dual currents, ${}^\star\bar{\mathcal{J}}_{+A} = \mathcal{T}_A{}^B \bar{\mathcal{J}}_{+B}$ and ${}^\star\mathcal{J}_{-A} = \mathcal{T}_A{}^B \mathcal{J}_{-B}$, in terms of the initial ones. These expressions produce the expressions for fermionic T-dual NS-R fields and R-R field strength. Expressions for fermionic T-dual metric and Kalb-Ramond field are obtained separately under some assumptions.

In this article we will not present explicitely the transformation of dilaton field, which demands quantum treatment.

2. Pure spinor action of type II action

The sigma model of pure spinor type II superstring action for type II superstring [32] is of the form

$$
\begin{aligned}
S &= \int d^2\xi \left[\partial_+\theta^\alpha A_{\alpha\beta}\partial_-\bar\theta^\beta + \partial_+\theta^\alpha A_{\alpha\mu}\bar\Pi^\mu + \Pi^\mu A_{\mu\alpha}\partial_-\bar\theta^\alpha + \Pi^\mu A_{\mu\nu}\bar\Pi^\nu \right. \\
&+ d_\alpha E^\alpha{}_\beta \partial_-\bar\theta^\beta + d_\alpha E^\alpha{}_\mu \bar\Pi^\mu + \partial_+\theta^\alpha E_\alpha{}^\beta \bar d_\beta + \Pi^\mu \bar E_\mu{}^\beta \bar d_\beta + d_\alpha \mathrm{P}^{\alpha\beta}\bar d_\beta \\
&+ \frac{1}{2}N^{\mu\nu}\Omega_{\mu\nu,\beta}\partial_-\bar\theta^\beta + \frac{1}{2}N^{\mu\nu}\Omega_{\mu\nu,\rho}\bar\Pi^\rho + \frac{1}{2}\partial_+\theta^\alpha\Omega_{\alpha,\mu\nu}\bar N^{\mu\nu} + \frac{1}{2}\Pi^\mu\Omega_{\mu,\nu\rho}\bar N^{\nu\rho} \\
&+ \left. \frac{1}{2}N^{\mu\nu}\bar C_{\mu\nu}{}^\beta \bar d_\beta + \frac{1}{2}d_\alpha C^\alpha{}_{\mu\nu}\bar N^{\mu\nu} + \frac{1}{4}N^{\mu\nu}S_{\mu\nu,\rho\sigma}\bar N^{\rho\sigma}\right] + S_\lambda + S_{\bar\lambda}\,, \qquad (1)
\end{aligned}
$$

where

$$
\Pi^\mu = \partial_+ x^\mu + \frac{1}{2}\theta^\alpha(\Gamma^\mu)_{\alpha\beta}\partial_+\theta^\beta\,, \quad \bar\Pi^\mu = \partial_- x^\mu + \frac{1}{2}\bar\theta^\alpha(\Gamma^\mu)_{\alpha\beta}\partial_-\bar\theta^\beta\,, \qquad (2)
$$

$$d_\alpha = \pi_\alpha - \frac{1}{2}(\Gamma_\mu\theta)_\alpha \left[\partial_+ x^\mu + \frac{1}{4}(\theta\Gamma_\mu\partial_+\theta)\right] ,$$
$$\bar{d}_\alpha = \bar{\pi}_\alpha - \frac{1}{2}(\Gamma_\mu\bar{\theta})_\alpha \left[\partial_- x^\mu + \frac{1}{4}(\bar{\theta}\Gamma_\mu\partial_-\bar{\theta})\right] , \quad (3)$$

$$N^{\mu\nu} = \frac{1}{2}w_\alpha(\Gamma^{[\mu\nu]})^\alpha{}_\beta\lambda^\beta , \quad \bar{N}^{\mu\nu} = \frac{1}{2}\bar{w}_\alpha(\Gamma^{[\mu\nu]})^\alpha{}_\beta\bar{\lambda}^\beta . \quad (4)$$

Type II superfields generally depends on x^μ, θ^α and $\bar{\theta}^\alpha$. The superfields $A_{\mu\nu}$, $\bar{E}_\mu{}^\alpha$, $E^\alpha{}_\mu$ and $\mathrm{P}^{\alpha\beta}$ are physical superfields, because their first components are supergravity fields. The fields $\Omega_{\mu,\nu\rho}(\Omega_{\mu\nu,\rho})$, $C^\alpha{}_{\mu\nu}(\bar{C}_{\mu\nu}{}^\alpha)$ and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical superfields. The rest fields are auxiliary superfirlds because they can be expressed in terms of the physical ones [32].

The world sheet Σ is parameterized by $\xi^m = (\xi^0 = \tau\,, \xi^1 = \sigma)$ and $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Superspace is spanned by bosonic coordinates x^μ ($\mu = 0, 1, 2, \dots, 9$) and fermionic ones θ^α and $\bar{\theta}^\alpha$ ($\alpha = 1, 2, \dots, 16$). The variables π_α and $\bar{\pi}_\alpha$ are canonically conjugated momenta to θ^α and $\bar{\theta}^\alpha$, respectively. The actions for pure spinors, S_λ and $S_{\bar{\lambda}}$, are free field actions

$$S_\lambda = \int d^2\xi w_\alpha \partial_- \lambda^\alpha , \quad S_{\bar{\lambda}} = \int d^2\xi \bar{w}_\alpha \partial_+ \bar{\lambda}^\alpha , \quad (5)$$

where λ^α and $\bar{\lambda}^\alpha$ are pure spinors and w_α and $\bar{w}_\alpha$ are their canonically conjugated momenta, respectively. The pure spinors satisfy so called pure spinor constraints

$$\lambda^\alpha(\Gamma^\mu)_{\alpha\beta}\lambda^\beta = \bar{\lambda}^\alpha(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0 . \quad (6)$$

3. Bosonic T-dualization

In this section we will introduce approximated action and then apply standard Buscher procedure on some subset of coordinates x^a. Then we will compare obtained result with the result following from double space formalism.

3.1. Simplification of action

The action (1) could be considered as an expansion in powers of θ^α and $\bar{\theta}^\alpha$. For computational simplicity, in the first step we neglect all terms in the action containing θ^α and $\bar{\theta}^\alpha$. As a consequence θ^α and $\bar{\theta}^\alpha$ terms disappear from $\Pi^\mu_\pm$, d_α and $\bar{d}_\alpha$ and in the solutions for physical superfields, just x-dependence of the supergravity fields survives.

Let us make T-dualization along some subset of bosonic coordinates x^a. So, we will assume that these directions are isometry ones. It essentially means that corresponding superfields (A_{ab}, $\bar{E}_a{}^\alpha$, $E^\alpha{}_a$, $\mathrm{P}^{\alpha\beta}$) should not depend on x^a. This assumption could be extended on all space-time directions x^μ which means that physical superfields are constant. According to [32], auxiliary superfields are zero, because all physical superfields are constant. Further, constant physical superfields means that their field strengths, $\Omega_{\mu,\nu\rho}(\Omega_{\mu\nu,\rho})$, $C^\alpha{}_{\mu\nu}(\bar{C}_{\mu\nu}{}^\alpha)$ and $S_{\mu\nu,\rho\sigma}$, are zero.

Background fields obey space-time field equations [36], which are some kind of consistency conditions. The equation (B.7) from this set of equations represents the backreaction of $\mathrm{P}^{\alpha\beta}$ on the metric $G_{\mu\nu}$. If we take constant dilaton Φ and constant antisymmetric NS-NS field $B_{\mu\nu}$ we obtain that

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R \sim (\mathrm{P}^{\alpha\beta})^2_{\mu\nu}\,. \tag{7}$$

If we choose the background field $\mathrm{P}^{\alpha\beta}$ to be constant, in general, we will have constant Ricci tensor which means that metric tensor is quadratic function of space-time coordinates. If one wants to cancel non-quadratic terms originating from back-reaction, additional conditions must be imposed on R-R field strength (see the first reference in [33]).

Taking into account above analysis and arguments, our approximation is realized in the following way

$$\Pi^\mu_\pm \to \partial_\pm x^\mu\,, \quad d_\alpha \to \pi_\alpha\,, \quad \bar{d}_\alpha \to \bar{\pi}_\alpha\,, \tag{8}$$

and physical superfields take the form

$$\begin{aligned} A_{\mu\nu} &= \kappa(\frac{1}{2}G_{\mu\nu} + B_{\mu\nu})\,, \quad E^\alpha_\nu = -\Psi^\alpha_\nu\,, \quad \bar{E}^\alpha_\mu = \bar{\Psi}^\alpha_\mu\,, \\ \mathrm{P}^{\alpha\beta} &= \frac{2}{\kappa}P^{\alpha\beta} = \frac{2}{\kappa}e^{\frac{\Phi}{2}}F^{\alpha\beta}\,, \end{aligned} \tag{9}$$

where $G_{\mu\nu}$ is metric tensor and $B_{\mu\nu}$ is antisymmetric NS-NS background field.

Consequently, the full action S is

$$\begin{aligned} S &= \kappa\int_\Sigma d^2\xi\left[\partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu + \frac{1}{4\pi\kappa}\Phi R^{(2)}\right] \\ &+ \int_\Sigma d^2\xi\left[-\pi_\alpha\partial_-(\theta^\alpha + \Psi^\alpha_\mu x^\mu) + \partial_+(\bar{\theta}^\alpha + \bar{\Psi}^\alpha_\mu x^\mu)\bar{\pi}_\alpha + \frac{2}{\kappa}\pi_\alpha P^{\alpha\beta}\bar{\pi}_\beta\right]\,, \end{aligned} \tag{10}$$

where

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}\,. \tag{11}$$

Actions S_λ and $S_{\bar{\lambda}}$ are decoupled from the rest and can be neglected in the further analysis.

3.2. Buscher T-dualization

Let some shift symmetry along x^a directions exists. The first step in Buscher procedure is that we localize the noticed shift symmetry. We substitute the ordinary derivatives with covariant ones, introducing gauge fields v_α^a. Then we add the term $\frac{1}{2}y_a F_{+-}^a$ to the Lagrangian in order to force the field strength F_{+-}^a to vanish and preserve equivalence between original and T-dual theories. Finally, we fix the gauge $x^a = 0$ and obtain gauge fixed action

$$\begin{aligned}
&S_{fix}(v_\pm^a, x^i, \theta^\alpha, \bar\theta^\alpha, \pi_\alpha, \bar\pi_\alpha) = \\
&\int_\Sigma d^2\xi \left[\kappa v_+^a \Pi_{+ab} v_-^b + \kappa v_+^a \Pi_{+aj}\partial_- x^j + \kappa \partial_+ x^i \Pi_{+ib} v_-^b + \right.\\
+ \quad & \kappa \partial_+ x^i \Pi_{+ij}\partial_- x^j + \frac{1}{4\pi}\Phi R^{(2)} - \pi_\alpha \Psi_b^\alpha v_-^b \\
+ \quad & v_+^a \bar\Psi_a^\alpha \bar\pi_\alpha - \pi_\alpha \partial_-(\theta^\alpha + \Psi_i^\alpha x^i) + \partial_+(\bar\theta^\alpha + \bar\Psi_i^\alpha x^i)\bar\pi_\alpha + \frac{1}{2\kappa} e^{\frac{\Phi}{2}}\pi_\alpha F^{\alpha\beta}\bar\pi_\beta \\
+ \quad & \left.\frac{\kappa}{2}(v_+^a \partial_- y_a - v_-^a \partial_+ y_a)\right] .
\end{aligned} \tag{12}$$

Varying the gauge fixed action with respect to the Lagrange multipliers y_a we get the solution for gauge fields in the form

$$v_\pm^a = \partial_\pm x^a \,, \tag{13}$$

which produces the initial action, while varying with respect to the gauge fields $v_\pm^a$ we have

$$v_\pm^a = -2\kappa \hat\theta_\pm^{ab}\Pi_{\mp bi}\partial_\pm x^i - \kappa \hat\theta_\pm^{ab}\partial_\pm y_b \pm 2\hat\theta_\pm^{ab}\Psi_{\pm b}^\alpha \pi_{\pm\alpha} \,. \tag{14}$$

Substituting $v_\pm^a$ in (12) we find

$$\begin{aligned}
&S_{fix}(y_a, x^i, \theta^\alpha, \bar\theta^\alpha, \pi_\alpha, \bar\pi_\alpha) = \\
&\int_\Sigma d^2\xi \left[\frac{\kappa^2}{2}\partial_+ y_a \hat\theta_-^{ab}\partial_- y_b + \kappa^2 \partial_+ y_a \hat\theta_-^{ab}\Pi_{+bj}\partial_- x^j - \kappa^2 \partial_+ x^i \Pi_{+ia}\hat\theta_-^{ab}\partial_- y_b \right.\\
+ \quad & \kappa \partial_+ x^i (\Pi_{+ij} - 2\kappa \Pi_{+ia}\hat\theta_-^{ab}\Pi_{+bj})\partial_- x^j + \frac{1}{4\pi}\Phi R^{(2)} \\
- \quad & \pi_\alpha \partial_-(\theta^\alpha + \Psi_i^\alpha x^i - 2\kappa \Psi_a^\alpha \hat\theta_-^{ab}\Pi_{+bj}x^j - \kappa \Psi_a^\alpha \hat\theta_-^{ab} y_b) \\
+ \quad & \partial_+(\bar\theta^\alpha + \bar\Psi_i^\alpha x^i + 2\kappa \bar\Psi_a^\alpha \hat\theta_+^{ab}\Pi_{-bj}x^j + \kappa \bar\Psi_a^\alpha \hat\theta_+^{ab} y_b)\bar\pi_\alpha \\
+ \quad & \left. 2\pi_\alpha \Psi_a^\alpha \hat\theta_-^{ab}\bar\Psi_b^\beta \bar\pi_\beta + \frac{1}{2\kappa} e^{\frac{\Phi}{2}}\pi_\alpha F^{\alpha\beta}\bar\pi_\beta \right] .
\end{aligned} \tag{15}$$

Combining two solutions for gauge fields (13) and (14) we obtain transformation law between initial x^a and T-dual coordinates y_a

$$\partial_+ ({}_aX)_{\hat\mu} = (\bar Q^{-1T})_{\hat\mu\nu}\partial_+ x^\nu + J_{+\hat\mu} \,, \quad \partial_- \, ({}_aX)_{\hat\mu} = (Q^{-1T})_{\hat\mu\nu}\partial_- x^\nu + J_{-\hat\mu} \,, \tag{16}$$

where we introduced the T-dual variables ${}_aX_{\hat\mu} = \{y_a,\, x^i\}$. For coordinates which contain both x^i and y_a we will use "hat" indices $\hat\mu, \hat\nu$. The matrices

$$Q^{\hat\mu\nu} = \begin{pmatrix} \kappa\hat\theta_+^{ab} & 0 \\ -2\kappa\Pi_{-ic}\hat\theta_+^{cb} & \delta^i_j \end{pmatrix}, \qquad \bar Q^{\hat\mu\nu} = \begin{pmatrix} \kappa\hat\theta_-^{ab} & 0 \\ -2\kappa\Pi_{+ic}\hat\theta_-^{cb} & \delta^i_j \end{pmatrix}, \tag{17}$$

and theirs inverse

$$Q^{-1}_{\mu\hat\nu} = \begin{pmatrix} 2\Pi_{-ab} & 0 \\ 2\Pi_{-ib} & \delta_i^j \end{pmatrix}, \qquad \bar Q^{-1}_{\mu\hat\nu} = \begin{pmatrix} 2\Pi_{+ab} & 0 \\ 2\Pi_{+ib} & \delta_i^j \end{pmatrix}, \tag{18}$$

perform T-dualization for vector indices. Here we introduced the currents $J_{\pm\hat\mu}$

$$J_{+\hat\mu} = \begin{pmatrix} J_{+a} \\ 0 \end{pmatrix}, \quad J_{-\hat\mu} = \begin{pmatrix} J_{-a} \\ 0 \end{pmatrix}, \tag{19}$$

where

$$J_{\pm\mu} = \pm\frac{2}{\kappa}\Psi^\alpha_{\pm\mu}\pi_{\pm\alpha}\,, \tag{20}$$

$$\Psi^\alpha_{+\mu} \equiv \Psi^\alpha_\mu\,, \quad \Psi^\alpha_{-\mu} \equiv \bar\Psi^\alpha_\mu\,, \quad \pi_{+\alpha} \equiv \pi_\alpha\,, \quad \pi_{-\alpha} \equiv \bar\pi_\alpha\,, \tag{21}$$

and $\hat\theta^{ab}_\pm$ is defined by the relation

$$\hat\theta^{ac}_\pm \Pi_{\mp cb} = \frac{1}{2\kappa}\delta^a{}_b\,. \tag{22}$$

Note that different chiralities transform with different matrices $Q^{\hat\mu\nu}$ and $\bar Q^{\hat\mu\nu}$. So, there are two types of T-dual vielbeins

$${}_ae^{\underline{a}\hat\mu} = e^{\underline{a}}{}_\nu (Q^T)^{\nu\hat\mu}\,, \qquad {}_a\bar e^{\underline{a}\hat\mu} = e^{\underline{a}}{}_\nu (\bar Q^T)^{\nu\hat\mu}\,, \tag{23}$$

with the same T-dual metric

$${}_aG^{\hat\mu\hat\nu} \equiv ({}_ae^T\eta\,{}_ae)^{\hat\mu\hat\nu} = (QGQ^T)^{\hat\mu\hat\nu} = {}_a\bar G^{\hat\mu\hat\nu} \equiv ({}_a\bar e^T\eta\,{}_a\bar e)^{\hat\mu\hat\nu} = (\bar QG\bar Q^T)^{\hat\mu\hat\nu}\,. \tag{24}$$

The two T-dual vielbeins are related by particular local Lorentz transformation

$${}_a\bar e^{\underline{a}\hat\mu} = \Lambda^{\underline{a}}{}_{\underline{b}}\,{}_ae^{\underline{b}\hat\mu}\,, \qquad \Lambda^{\underline{a}}{}_{\underline{b}} = e^{\underline{a}}{}_\mu (Q^{-1}\bar Q)^{T\mu}{}_\nu (e^{-1})^\nu{}_{\underline{b}}\,. \tag{25}$$

From (17) and (18) we have

$$(Q^{-1}\bar Q)^{T\mu}{}_\nu = \begin{pmatrix} \delta^a{}_b + 2\kappa\hat\theta^{ac}_+ G_{cb} & 2\kappa\hat\theta^{ac}_+ G_{cj} \\ 0 & \delta^i_j \end{pmatrix}, \tag{26}$$

which produces

$$\Lambda^{\underline{a}}{}_{\underline{b}} = \delta^{\underline{a}}{}_{\underline{b}} - 2\omega^{\underline{a}}{}_{\underline{b}}\,, \qquad \omega^{\underline{a}}{}_{\underline{b}} = -\kappa e^{\underline{a}}{}_a\hat\theta^{ab}_+ (e^T)_b{}^{\underline{c}}\eta_{\underline{cb}}\,. \tag{27}$$

It satisfies definition of Lorentz transformations

$$\Lambda^T \eta \Lambda = \eta \quad \Longrightarrow \quad \det \Lambda^{\underline{a}}{}_{\underline{b}} = \pm 1\,, \tag{28}$$

and it holds $\det \Lambda^{\underline{a}}{}_{\underline{b}} = (-1)^d$, where d is the number of dimensions along which we perform T-duality.

Existence of the local Lorentz transformation which connects two sets of vielbeins means that in T-dual picture we can take that, for example, nonbar variables remain the same while bar variables must be multiplied by matrix ${}_a\Omega$, which is a spinorial representation of that local Lorentz symmetry. Skipping technical details explained in details in [37], we give here the final form of the matrix ${}_a\Omega$

$${}_a\Omega \; = \sqrt{\prod_{i=1}^{d} G_{a_i a_i}}\; {}_a\Gamma \, (i\,\Gamma^{11})^d\,, \tag{29}$$

where

$${}_a\Gamma \; = (i)^{\frac{d(d-1)}{2}} \prod_{i=1}^{d} \Gamma^{a_i} = (i)^{\frac{d(d-1)}{2}}\, \Gamma^{a_1}\Gamma^{a_2}\cdots\Gamma^{a_d}\,. \tag{30}$$

It is easy to check that ${}_a\Omega^2 = 1$.

Let we introduce proper fermionic variables

$${}_a\theta^\alpha = \theta^\alpha\,, \quad {}_a\pi_\alpha = \pi_\alpha\,, \quad {}^\bullet\bar\theta^\alpha \equiv {}_a\Omega^\alpha{}_\beta\, {}_a\bar\theta^\beta\,, \quad {}^\bullet\bar\pi_\alpha \equiv {}_a\Omega_\alpha{}^\beta\, {}_a\bar\pi_\beta\,. \tag{31}$$

Using the action (15) and proper fermionic variables, we read T-dual background fields

$${}_a\Pi^{ab}_\pm = \frac{\kappa}{2}\hat\theta^{ab}_\mp\,, \tag{32}$$

$${}_a\Pi_{\pm i}{}^a = -\kappa \Pi_{\pm ib}\hat\theta^{ba}_\mp\,, \quad {}_a(\Pi_\pm)^a{}_i = \kappa\hat\theta^{ab}_\mp \Pi_{\pm bi}\,, \tag{33}$$

$${}_a\Pi_{\pm ij} = \Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\hat\theta^{ab}_\mp \Pi_{\pm bj}\,, \tag{34}$$

$${}_a\Psi^{\alpha a} = \kappa\hat\theta^{ab}_+ \Psi^\alpha_b\,, \quad {}_a\bar\Psi^{\alpha a} = \kappa\, {}_a\Omega^\alpha{}_\beta\, \hat\theta^{ab}_- \bar\Psi^\beta_b\,, \tag{35}$$

$${}_a\Psi^\alpha_i = \Psi^\alpha_i - 2\kappa\Pi_{-ib}\hat\theta^{ba}_+\Psi^\alpha_a\,, \quad {}_a\bar\Psi^\alpha_i = {}_a\Omega^\alpha{}_\beta(\bar\Psi^\beta_i - 2\kappa\Pi_{+ib}\hat\theta^{ba}_-\bar\Psi^\beta)\,, \tag{36}$$

$$e^{\frac{{}_a\Phi}{2}}\, {}_aF^{\alpha\beta} = (e^{\frac{\Phi}{2}}\, F^{\alpha\gamma} + 4\kappa\Psi^\alpha_a\hat\theta^{ab}_-\bar\Psi^\gamma_b)\, {}_a\Omega_\gamma{}^\beta\,. \tag{37}$$

3.3. Double space

Above expressions for T-dual background fields in the case of full T-dualization are

$${}^\star\Pi^{\mu\nu}_\pm \equiv {}^\star B^{\mu\nu} \pm \frac{1}{2}{}^\star G^{\mu\nu} = \frac{\kappa}{2}\Theta^{\mu\nu}_\mp\,, \tag{38}$$

$${}^\star\Psi^{\alpha\mu} = \kappa\Theta^{\mu\nu}_+\Psi^\alpha_\nu\,, \quad {}^\star\bar\Psi^{\alpha\mu} = \kappa\, {}^\star\Omega^\alpha{}_\beta\, \Theta^{\mu\nu}_-\bar\Psi^\beta_\nu\,, \tag{39}$$

$$e^{\frac{\star\Phi}{2}}\,{}^{\star}F^{\alpha\beta} = (e^{\frac{\Phi}{2}}F^{\alpha\gamma} + 4\kappa\Psi^{\alpha}_{\mu}\Theta^{\mu\nu}_{-}\bar{\Psi}^{\gamma}_{\nu})^{\star}\Omega_{\gamma}{}^{\beta}\,. \tag{40}$$

Here we use the notation

$$G^{E}_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}\,, \quad \Theta^{\mu\nu} = -\frac{2}{\kappa}(G^{-1}_{E}BG^{-1})^{\mu\nu}\,, {}^{\star}\Omega = -\Gamma^{11}, \tag{41}$$

and

$$\Theta^{\mu\nu}_{\pm} = -\frac{2}{\kappa}(G^{-1}_{E}\Pi_{\pm}G^{-1})^{\mu\nu} = \Theta^{\mu\nu} \mp \frac{1}{\kappa}(G^{-1}_{E})^{\mu\nu}\,. \tag{42}$$

In this case the T-dual transformation laws (16) obtain the form

$$\partial_{\pm}x^{\mu} \cong -\kappa\Theta^{\mu\nu}_{\pm}\partial_{\pm}y_{\nu} + \kappa\Theta^{\mu\nu}_{\pm}J_{\pm\nu}\,, \qquad \partial_{\pm}y_{\mu} \cong -2\Pi_{\mp\mu\nu}\partial_{\pm}x^{\nu} + J_{\pm\mu}\,. \tag{43}$$

In terms of double coordinates

$$Z^{M} = \begin{pmatrix} x^{\mu} \\ y_{\mu} \end{pmatrix}, \tag{44}$$

the relations (16) are replaced by one

$$\partial_{\pm}Z^{M} \cong \pm\Omega^{MN}\left(\mathcal{H}_{NP}\partial_{\pm}Z^{P} + J_{\pm N}\right), \tag{45}$$

where the matrix $\mathcal{H}_{MN}$ is generalized metric and has the form

$$\mathcal{H}_{MN} = \begin{pmatrix} G^{E}_{\mu\nu} & -2\,B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}\,B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}. \tag{46}$$

The double current $J_{\pm M}$ is defined as

$$J_{\pm M} = \begin{pmatrix} 2(\Pi_{\pm}G^{-1})_{\mu}{}^{\nu}J_{\pm\nu} \\ -(G^{-1})^{\mu\nu}J_{\pm\nu} \end{pmatrix}, \tag{47}$$

and

$$\Omega^{MN} = \begin{pmatrix} 0 & 1_{D} \\ 1_{D} & 0 \end{pmatrix}, \tag{48}$$

is constant symmetric matrix. Here 1_D denotes the identity operator in D dimensions.

It is known that equations of motion of initial theory are Bianchi identities in T-dual picture and vice versa [9, 13, 16, 38]. From Bianchi identity

$$\partial_{+}\partial_{-}Z^{M} - \partial_{-}\partial_{+}Z^{M} = 0\,, \tag{49}$$

and relation (45), we obtain the consistency condition

$$\partial_{+}\left[\mathcal{H}_{MN}\partial_{-}Z^{N} + J_{-M}\right] + \partial_{-}\left[\mathcal{H}_{MN}\partial_{+}Z^{N} + J_{+M}\right] = 0\,. \tag{50}$$

The equation (50) is equation of motion of the following action

$$S=\frac{\kappa}{4}\int d^2\xi\left[\partial_+Z^M\mathcal{H}_{MN}\partial_-Z^N+\partial_+Z^MJ_{-M}+J_{+M}\partial_-Z^M+L(\pi_\alpha,\bar\pi_\alpha)\right], \tag{51}$$

where $L(\pi_\alpha,\bar\pi_\alpha)$ is arbitrary functional of fermionic momenta.

Let us split coordinate index μ into a and i ($a=0,\cdots,d-1$, $i=d,\cdots,D-1$) and denote T-dualization along direction x^a and y_a as

$$\mathcal{T}^a=T^a\circ T_a\,,\quad T^a\equiv T^0\circ T^1\circ\cdots\circ T^{d-1}\,,\quad T_a\equiv T_0\circ T_1\circ\cdots\circ T_{d-1}\,, \tag{52}$$

where $\circ$ marks the operation of composition of T-dualizations. Permutation of the initial coordinates x^a with its T-dual y_a we realize by multiplying double space coordinate by the constant symmetric matrix $(\mathcal{T}^a)^M{}_N$

$${}_aZ^M\equiv\begin{pmatrix}y_a\\x^i\\x^a\\y_i\end{pmatrix}=(\mathcal{T}^a)^M{}_NZ^N\equiv\begin{pmatrix}0&0&1_a&0\\0&1_i&0&0\\1_a&0&0&0\\0&0&0&1_i\end{pmatrix}\begin{pmatrix}x^a\\x^i\\y_a\\y_i\end{pmatrix}, \tag{53}$$

where 1_a and 1_i are identity operators in the subspaces spanned by x^a and x^i, respectively. We demand that double T-dual coordinate ${}_aZ^M$ satisfy the T-duality transformations of the form as initial one Z^M (45)

$$\partial_\pm\,{}_aZ^M\cong\pm\,\Omega^{MN}\left({}_a\mathcal{H}_{NK}\,\partial_\pm\,{}_aZ^K+{}_aJ_{\pm N}\right). \tag{54}$$

From this relation we find the T-dual generalized metric

$${}_a\mathcal{H}_{MN}=(\mathcal{T}^a)_M{}^K\mathcal{H}_{KL}(\mathcal{T}^a)^L{}_N\,, \tag{55}$$

and T-dual current

$${}_aJ_{\pm M}=(\mathcal{T}^a)_M{}^NJ_{\pm N}\,. \tag{56}$$

Demanding that the T-dual generalized metric ${}_a\mathcal{H}_{MN}$ has the same form as the initial one $\mathcal{H}_{MN}$ (46)

$${}_a\mathcal{H}_{MN}=\begin{pmatrix}{}_aG_E^{\mu\nu}&-2({}_aB\,{}_aG^{-1})^\mu{}_\nu\\2({}_aG^{-1}\,{}_aB)_\mu{}^\nu&({}_aG^{-1})_{\mu\nu}\end{pmatrix}, \tag{57}$$

we obtain the T-dual NS-NS background fields

$${}_a\Pi_\pm^{ab}=\tfrac{\kappa}{2}\hat\theta_\mp^{ab}\,,\qquad {}_a\Pi^a_{\pm i}=\kappa\hat\theta_\mp^{ab}\Pi_{\pm bi}\,, \tag{58}$$

$${}_a\Pi_{\pm i}{}^a=-\kappa\Pi_{\pm ib}\hat\theta_\mp^{ba}\,,\quad {}_a\Pi_{\pm ij}=\Pi_{\pm ij}-2\kappa\Pi_{\pm ia}\hat\theta_\mp^{ab}\Pi_{\pm bj}\,, \tag{59}$$

which are in full agreement with those from the Refs.[6, 14].

The T-dual current ${}_aJ_{\pm M}$ (56) should have the same form as initial one (47) but in terms of the T-dual background fields

$$\begin{pmatrix} 2({}_a\Pi_{\pm}\, {}_aG^{-1})^a{}_b\, ({}_aJ)^b_{\pm} + 2({}_a\Pi_{\pm}\, {}_aG^{-1})^{ai}\, ({}_aJ)_{\pm i} \\ 2({}_a\Pi_{\pm}\, {}_aG^{-1})_{ia}\, ({}_aJ)^a_{\pm} + 2({}_a\Pi_{\pm}\, {}_aG^{-1})_i{}^j\, ({}_aJ)_{\pm j} \\ -({}_aG^{-1})_{ab}\, ({}_aJ)^b_{\pm} - ({}_aG^{-1})_a{}^i\, ({}_aJ)_{\pm i} \\ -({}_aG^{-1})^i{}_a\, ({}_aJ)^a_{\pm} - ({}_aG^{-1})^{ij}\, ({}_aJ)_{\pm j} \end{pmatrix} = \begin{pmatrix} -(G^{-1})^{a\mu} J_{\pm\mu} \\ 2(\Pi_{\pm}G^{-1})_i{}^{\mu} J_{\pm\mu} \\ 2(\Pi_{\pm}G^{-1})_a{}^{\mu} J_{\pm\mu} \\ -(G^{-1})^{i\mu} J_{\pm\mu} \end{pmatrix}. \tag{60}$$

From the lower D components of the above equation, after straightforward calculation we get

$${}_a\Psi^{\alpha a} = \kappa\hat{\theta}^{ab}_{+}\Psi^{\alpha}_b\,, \quad {}_a\bar{\Psi}^{\alpha a} = \kappa\, {}_a\Omega^{\alpha}{}_{\beta}\hat{\theta}^{ab}_{-}\bar{\Psi}^{\beta}_b\,. \tag{61}$$

$${}_a\Psi^{\alpha}_i = \Psi^{\alpha}_i - 2\kappa\Pi_{-ib}\hat{\theta}^{ba}_{+}\Psi^{\alpha}_a\,, \quad {}_a\bar{\Psi}^{\alpha}_i = {}_a\Omega^{\alpha}{}_{\beta}(\bar{\Psi}^{\beta}_i - 2\kappa\Pi_{+ib}\hat{\theta}^{ba}_{-}\bar{\Psi}^{\beta}_a)\,. \tag{62}$$

which is in full agreement with results obtained applying standard Buscher procedure. The upper D components of Eq.(60) produce the same result for T-dual background fields.

The R-R field strength $F^{\alpha\beta}$ appears in the action (10) coupled with fermionic momenta π_α and $\bar{\pi}_\alpha$ along which we do not perform T-dualization. Let us suppose that fermionic term $L(\pi_\alpha, \bar{\pi}_\alpha)$ (51) in the form

$$L = e^{\frac{\Phi}{2}}\, \pi_\alpha\, F^{\alpha\beta}\bar{\pi}_\beta + e^{\frac{{}_a\Phi}{2}}\, {}_a\pi_\alpha\, {}_aF^{\alpha\beta}\, {}_a\bar{\pi}_\beta \equiv \mathcal{L} + {}_a\mathcal{L}\,, \tag{63}$$

for some $F^{\alpha\beta}$ and ${}_aF^{\alpha\beta}$. This term should be invariant under T-dual transformation

$${}_a\mathcal{L} = \mathcal{L} + \Delta\mathcal{L}\,. \tag{64}$$

Taking into account the fact that two successive T-dualization are identity transformation, we obtain that the T-dual R-R field strength has the form

$$e^{\frac{{}_a\Phi}{2}}\, {}_aF^{\alpha\beta} = (e^{\frac{\Phi}{2}} F^{\alpha\gamma} + c\Psi^{\alpha}_a\hat{\theta}^{ab}_{-}\bar{\Psi}^{\gamma}_b)\, {}_a\Omega_{\gamma}{}^{\beta}\,. \tag{65}$$

For $c = 4\kappa$ we obtain the agreement with the expression (37).

4. Fermionic T-dualization

4.1. Action

We start with the action (1). In order to perform fermionic T-duality we must avoid explicit dependence of background fields on the fermionic coordinates θ^α and $\bar{\theta}^\alpha$ and allow only dependence on the σ and τ derivatives of these coordinates. This assumption produces that the auxiliary superfields are zero what can be seen from Eq.(5.5) of Ref.[32].

We choose that $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, Ψ^{α}_{μ} and $\bar{\Psi}^{\alpha}_{\mu}$ are constant, and corresponding field strengths, $\Omega_{\mu,\nu\rho}(\Omega_{\mu\nu,\rho})$, $C^{\alpha}{}_{\mu\nu}(\bar{C}_{\mu\nu}{}^{\alpha})$ and $S_{\mu\nu,\rho\sigma}$, are zero. The only nontrivial contribution of the quadratic terms in equations of motion (see

[36]) comes from constant field strength $\mathrm{P}^{\alpha\beta}$. In order to analyze this issue we will use relations from Eq.(3.6) of Ref.[32] labeled by $(\frac{1}{2}, \frac{3}{2}, \frac{3}{2})$

$$D_\alpha \mathrm{P}^{\beta\gamma} - \frac{1}{4}(\Gamma^{\mu\nu})_\alpha{}^\beta \bar{C}_{\mu\nu}{}^\gamma = 0\,, \quad \bar{D}_\alpha \mathrm{P}^{\beta\gamma} - \frac{1}{4}(\Gamma^{\mu\nu})_\alpha{}^\gamma C^\beta{}_{\mu\nu} = 0\,. \tag{66}$$

Here

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\Gamma^\mu\theta)_\alpha \frac{\partial}{\partial x^\mu}\,, \quad \bar{D}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{1}{2}(\Gamma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}\,, \tag{67}$$

are superspace covariant derivatives and $C^\alpha{}_{\mu\nu}$ and $\bar{C}_{\mu\nu}{}^\alpha$ are field strengths for gravitino fields Ψ^α_μ and $\bar{\Psi}^\alpha_\mu$, respectively. In order to perform fermionic T-dualization along all fermionic directions, θ^α and $\bar{\theta}^\alpha$, we assume that they are Killing spinors which means

$$\frac{\partial \mathrm{P}^{\beta\gamma}}{\partial\theta^\alpha} = \frac{\partial \mathrm{P}^{\beta\gamma}}{\partial\bar{\theta}^\alpha} = 0\,. \tag{68}$$

From the equations (66) it follows

$$(\Gamma^\mu)_{\alpha\delta}\partial_\mu \mathrm{P}^{\beta\gamma} = 0\,. \tag{69}$$

Our choice of constant $\mathrm{P}^{\alpha\beta}$ is consistent with this condition.

Under these assumptions the final form of the action is given by the expression (10). The fermionic part of the action (10) has the form of the first order theory. On the equations of motion for fermionic momenta π_α and $\bar{\pi}_\alpha$,

$$\pi_\alpha = -\frac{\kappa}{2}\partial_+\left(\bar{\theta}^\beta + \bar{\Psi}^\beta_\mu x^\mu\right)(P^{-1})_{\beta\alpha}\,, \quad \bar{\pi}_\alpha = \frac{\kappa}{2}(P^{-1})_{\alpha\beta}\partial_-\left(\theta^\beta + \Psi^\beta_\mu x^\mu\right)\,, \tag{70}$$

the action gets the form

$$\begin{aligned} & S(\partial_\pm x, \partial_-\theta, \partial_+\bar{\theta}) = \kappa\int_\Sigma d^2\xi\,\partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu + \frac{1}{4\pi}\int_\Sigma d^2\xi\,\Phi R^{(2)} \\ + & \frac{\kappa}{2}\int_\Sigma d^2\xi\,\partial_+\left(\bar{\theta}^\alpha + \bar{\Psi}^\alpha_\mu x^\mu\right)(P^{-1})_{\alpha\beta}\partial_-\left(\theta^\beta + \Psi^\beta_\nu x^\nu\right) \\ = & \kappa\int_\Sigma d^2\xi\,\partial_+ x^\mu\left[\Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu (P^{-1})_{\alpha\beta}\Psi^\beta_\nu\right]\partial_- x^\nu + \frac{1}{4\pi}\int_\Sigma d^2\xi\,\Phi R^{(2)} \\ + & \frac{\kappa}{2}\int_\Sigma d^2\xi\left[\partial_+\bar{\theta}^\alpha (P^{-1})_{\alpha\beta}\partial_-\theta^\beta + \partial_+\bar{\theta}^\alpha (P^{-1}\Psi)_{\alpha\mu}\partial_- x^\nu\right. \\ + & \left.\partial_+ x^\mu (\bar{\Psi}P^{-1})_{\mu\alpha}\partial_-\theta^\alpha\right]\,. \end{aligned} \tag{71}$$

We notice that theory has a local symmetry

$$\delta\theta^\alpha = \varepsilon^\alpha(\sigma^+)\,, \quad \delta\bar\theta^\alpha = \bar\varepsilon^\alpha(\sigma^-)\,, \quad (\sigma^\pm = \tau \pm \sigma)\,. \tag{72}$$

The corresponding BRST transformations are

$$s\theta^\alpha = c^\alpha(\sigma^+)\,, \quad s\bar\theta^\alpha = \bar c^\alpha(\sigma^-)\,, \tag{73}$$

where for each gauge parameter $\varepsilon^\alpha(\sigma^+)$ and $\bar\varepsilon^\alpha(\sigma^-)$ we introduced the ghost fields $c^\alpha(\sigma^+)$ and $\bar c^\alpha(\sigma^-)$, respectively. Here s denotes BRST nilpotent operator.

To fix gauge freedom we introduce gauge fermion with ghost number -1

$$\Psi = \frac{\kappa}{2}\int d^2\xi \left[\bar C_\alpha\left(\partial_+\theta^\alpha + \frac{\alpha^{\alpha\beta}}{2}b_{+\beta}\right) + \left(\partial_-\bar\theta^\alpha + \frac{1}{2}\bar b_{-\beta}\alpha^{\beta\alpha}\right)C_\alpha\right]\,, \tag{74}$$

where $\alpha^{\alpha\beta}$ is arbitrary non singular matrix, $\bar C_\alpha$ and C_α are antighost fields, while $b_{+\alpha}$ and $\bar b_{-\alpha}$ are Nakanishi-Lautrup auxillary fields which satisfy

$$sC_\alpha = b_{+\alpha}\,, \quad s\bar C_\alpha = \bar b_{-\alpha}\,, \quad sb_{+\alpha} = 0 \quad s\bar b_{-\alpha} = 0\,. \tag{75}$$

BRST transformation of gauge fermion Ψ produces the gauge fixed and Fadeev-Popov action

$$\begin{aligned} s\Psi &= S_{gf} + S_{FP}\,, \\ S_{gf} &= \frac{\kappa}{2}\int d^2\xi \left[\bar b_{-\alpha}\partial_+\theta^\alpha + \partial_-\bar\theta^\alpha b_{+\alpha} + \bar b_{-\alpha}\alpha^{\alpha\beta}b_{+\beta}\right]\,, \\ S_{FD} &= \frac{\kappa}{2}\int d^2\xi \left[\bar C_\alpha\partial_+c^\alpha + (\partial_-\bar c^\alpha)C_\alpha\right]\,. \end{aligned} \tag{76}$$

The Fadeev-Popov action is decoupled from the rest and, consequently, it can be omitted in further analysis. On the equations of motion for b-fields

$$b_{+\alpha} = -(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\alpha\,, \quad \bar b_{-\alpha} = -\partial_-\bar\theta^\beta(\alpha^{-1})_{\beta\alpha}\,, \tag{77}$$

we obtain the final form of the BRST gauge fixed action

$$S_{gf} = -\frac{\kappa}{2}\int d^2\xi\partial_-\bar\theta^\alpha(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta\,. \tag{78}$$

4.2. Fermionic T-duality using Buscher rules

As in the bosonic case we introduce gauge fields $v_\pm^\alpha$ and $\bar v_\pm^\alpha$ and replace ordinary world-sheet derivatives with covariant ones

$$\partial_\pm\theta^\alpha \to D_\pm\theta^\alpha \equiv \partial_\pm\theta^\alpha + v_\pm^\alpha\,, \quad \partial_\pm\bar\theta^\alpha \to D_\pm\bar\theta^\alpha \equiv \partial_\pm\bar\theta^\alpha + \bar v_\pm^\alpha\,. \tag{79}$$

In order to make the fields $v_\pm^\alpha$ and $\bar{v}_\pm^\alpha$ to be unphysical we add the following terms in the action

$$S_{gauge}(\vartheta, v_\pm, \bar{\vartheta}, \bar{v}_\pm) = \frac{1}{2}\kappa \int_\Sigma d^2\xi \bar{\vartheta}_\alpha(\partial_+ v_-^\alpha - \partial_- v_+^\alpha) + \frac{1}{2}\kappa \int_\Sigma d^2\xi (\partial_+ \bar{v}_-^\alpha - \partial_- \bar{v}_+^\alpha)\vartheta_\alpha \,, \tag{80}$$

where ϑ_α and $\bar{\vartheta}_\alpha$ are Lagrange multipliers. The full gauge invariant action is of the form

$$\begin{aligned} & S_{inv}(x, \theta, \bar{\theta}, \vartheta, \bar{\vartheta}, v_\pm, \bar{v}_\pm) = S(\partial_\pm x, D_-\theta, D_+\bar{\theta}) \\ + \;& S_{gf}(D_-\theta, D_+\bar{\theta}) + S_{gauge}(\vartheta, \bar{\vartheta}, v_\pm, \bar{v}_\pm)\,. \end{aligned} \tag{81}$$

Fixing θ^α and $\bar{\theta}^\alpha$ to zero we obtain the gauge fixed action

$$\begin{aligned} S_{fix} &= \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \left[\Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta}\Psi_\nu^\beta\right]\partial_- x^\nu + \frac{1}{4\pi}\int_\Sigma d^2\xi \Phi R^{(2)} \\ &+ \frac{\kappa}{2}\int_\Sigma \Big[\bar{v}_+^\alpha (P^{-1})_{\alpha\beta} v_-^\beta + \bar{v}_+^\alpha (P^{-1})_{\alpha\beta}\Psi_\nu^\beta \partial_- x^\nu \\ + \;& \partial_+ x^\mu \bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} v_-^\beta - \bar{v}_-^\alpha (\alpha^{-1})_{\alpha\beta} v_+^\beta\Big] \\ &+ \frac{\kappa}{2}\int_\Sigma d^2\xi \bar{\vartheta}_\alpha (\partial_+ v_-^\alpha - \partial_- v_+^\alpha) + \frac{\kappa}{2}\int_\Sigma d^2\xi (\partial_+ \bar{v}_-^\alpha - \partial_- \bar{v}_+^\alpha)\vartheta_\alpha\,. \end{aligned} \tag{82}$$

Varying the above action with respect to the Lagrange multipliers we obtain the initial action (71) because

$$\partial_+ v_-^\alpha - \partial_- v_+^\alpha = 0 \Longrightarrow v_\pm^\alpha = \partial_\pm \theta^\alpha \,, \quad \partial_+ \bar{v}_-^\alpha - \partial_- \bar{v}_+^\alpha = 0 \Longrightarrow \bar{v}_\pm^\alpha = \partial_\pm \bar{\theta}^\alpha \,. \tag{83}$$

On the other side, the equations of motion for $v_\pm^\alpha$ and $\bar{v}_\pm^\alpha$ give

$$\bar{v}_-^\alpha = \partial_- \bar{\vartheta}_\beta \alpha^{\beta\alpha}\,, \quad \bar{v}_+^\alpha = \partial_+ \bar{\vartheta}_\beta P^{\beta\alpha} - \partial_+ x^\mu \bar{\Psi}_\mu^\alpha \,, \tag{84}$$

$$v_+^\alpha = -\alpha^{\alpha\beta}\partial_+\vartheta_\beta \,, \quad v_-^\alpha = -P^{\alpha\beta}\partial_-\vartheta_\beta - \Psi_\mu^\alpha \partial_- x^\mu \,. \tag{85}$$

Substituting these expressions in the action S_{fix} we obtain the fermionic T-dual action

$$\begin{aligned} & {}^\star S(\partial_\pm x, \partial_-\vartheta, \partial_+\bar{\vartheta}) = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu + \frac{1}{4\pi}\int_\Sigma d^2\xi \; {}^\star\Phi R^{(2)}\,, \\ & + \frac{\kappa}{2}\int_\Sigma d^2\xi \left[\partial_+\bar{\vartheta}_\alpha P^{\alpha\beta}\partial_-\vartheta_\beta\right. \\ - \;& \left.\partial_+ x^\mu \bar{\Psi}_\mu^\alpha \partial_-\vartheta_\alpha + \partial_+\bar{\vartheta}_\alpha \Psi_\mu^\alpha \partial_- x^\mu - \partial_-\bar{\vartheta}_\alpha \alpha^{\alpha\beta}\partial_+\vartheta_\beta\right]\,. \end{aligned} \tag{86}$$

We read fermionic T-dual background fields

$${}^\star\Psi_{\alpha\mu} = (P^{-1}\Psi)_{\alpha\mu}\,, \; {}^\star\bar{\Psi}_{\mu\alpha} = -(\bar{\Psi}P^{-1})_{\mu\alpha}\,, \tag{87}$$

$$
{}^{\star}P_{\alpha\beta} = (P^{-1})_{\alpha\beta}\,, \quad {}^{\star}\alpha_{\alpha\beta} = (\alpha^{-1})_{\alpha\beta}\,. \tag{88}
$$

From the condition

$$
{}^{\star}\Pi_{+\mu\nu} + \frac{1}{2}{}^{\star}\bar{\Psi}_{\alpha\mu}\,({}^{\star}P^{-1})^{\alpha\beta}\,{}^{\star}\Psi_{\beta\nu} = \Pi_{+\mu\nu}\,, \tag{89}
$$

the fermionic T-dual metric and Kalb-Ramond field are

$$
\begin{aligned}
{}^{\star}G_{\mu\nu} &= G_{\mu\nu} + \frac{1}{2}\left[(\bar{\Psi}P^{-1}\Psi)_{\mu\nu} + (\bar{\Psi}P^{-1}\Psi)_{\nu\mu}\right]\,,\\
{}^{\star}B_{\mu\nu} &= B_{\mu\nu} + \frac{1}{4}\left[(\bar{\Psi}P^{-1}\Psi)_{\mu\nu} - (\bar{\Psi}P^{-1}\Psi)_{\nu\mu}\right]\,.
\end{aligned} \tag{90}
$$

We obtain T-dual transformation laws combining the different solutions of equations of motion for $v^\alpha_\pm$ and $\bar{v}^\alpha_\pm$ (83) and (84)-(85)

$$
\partial_-\theta^\alpha \cong -P^{\alpha\beta}\partial_-\vartheta_\beta - \Psi^\alpha_\mu\partial_-x^\mu\,, \quad \partial_+\bar{\theta}^\alpha \cong \partial_+\bar{\vartheta}_\beta P^{\beta\alpha} - \partial_+x^\mu\bar{\Psi}^\alpha_\mu\,, \tag{91}
$$

$$
\partial_+\theta^\alpha \cong -\alpha^{\alpha\beta}\partial_+\vartheta_\beta\,, \quad \partial_-\bar{\theta}^\alpha \cong \partial_-\bar{\vartheta}_\beta\alpha^{\beta\alpha}\,. \tag{92}
$$

From these relations we can obtain inverse transformation rules

$$
\begin{aligned}
\partial_-\vartheta_\alpha &\cong -(P^{-1})_{\alpha\beta}\partial_-\theta^\beta - (P^{-1})_{\alpha\beta}\Psi^\beta_\mu\partial_-x^\mu\,,\\
\partial_+\bar{\vartheta}_\alpha &\cong \partial_+\bar{\theta}^\beta(P^{-1})_{\beta\alpha} + \partial_+x^\mu\bar{\Psi}^\beta_\mu(P^{-1})_{\beta\alpha}\,,
\end{aligned} \tag{93}
$$

$$
\partial_+\vartheta_\alpha \cong -(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta\,, \quad \partial_-\bar{\vartheta}_\alpha \cong \partial_-\bar{\theta}^\beta(\alpha^{-1})_{\beta\alpha}\,. \tag{94}
$$

4.3. Fermionic T-dualization using double space

Now we will extend the meaning of the double space and double both fermionic coordinate as

$$
\Theta^A = \begin{pmatrix} \theta^\alpha \\ \vartheta_\alpha \end{pmatrix}\,, \quad \bar{\Theta}^A = \begin{pmatrix} \bar{\theta}^\alpha \\ \bar{\vartheta}_\alpha \end{pmatrix}\,. \tag{95}
$$

The transformation laws, (91)-(94), can be rewritten in the form

$$
\partial_-\Theta^A \cong -\Omega^{AB}\left[\mathcal{F}_{BC}\partial_-\Theta^C + \mathcal{J}_{-B}\right]\,, \quad \partial_+\bar{\Theta}^A \cong \left[\partial_+\bar{\Theta}^C\mathcal{F}_{CB} + \bar{\mathcal{J}}_{+B}\right]\Omega^{BA}\,, \tag{96}
$$

$$
\partial_+\Theta^A \cong -\Omega^{AB}\mathcal{A}_{BC}\partial_+\Theta^C\,, \quad \partial_-\bar{\Theta}^A \cong \partial_-\bar{\Theta}^C\mathcal{A}_{CB}\Omega^{BA}\,, \tag{97}
$$

where "fermionic generalized metric" $\mathcal{F}_{AB}$ has the form

$$
\mathcal{F}_{AB} = \begin{pmatrix} (P^{-1})_{\alpha\beta} & 0 \\ 0 & P^{\gamma\delta} \end{pmatrix}\,, \tag{98}
$$

and

$$\mathcal{A}_{AB} = \begin{pmatrix} (\alpha^{-1})_{\alpha\beta} & 0 \\ 0 & \alpha^{\gamma\delta} \end{pmatrix} . \tag{99}$$

The double currents, $\bar{\mathcal{J}}_{+A}$ and $\mathcal{J}_{-A}$, are of the form

$$\bar{\mathcal{J}}_{+A} = \begin{pmatrix} (\bar{\Psi}P^{-1})_{\mu\alpha}\partial_+ x^\mu \\ -\bar{\Psi}^\alpha_\mu \partial_+ x^\mu \end{pmatrix} , \quad \mathcal{J}_{-A} = \begin{pmatrix} (P^{-1}\Psi)_{\alpha\mu}\partial_- x^\mu \\ \Psi^\alpha_\mu \partial_- x^\mu \end{pmatrix} . \tag{100}$$

Let us introduce the permutation matrix

$$\mathcal{T}^A{}_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{101}$$

so that double T-dual coordinates are

$${}^\star\Theta^A = \mathcal{T}^A{}_B \Theta^B , \quad {}^\star\bar{\Theta}^A = \mathcal{T}^A{}_B \bar{\Theta}^B . \tag{102}$$

As in the case of bosonic T-dualization, from demand that T-dual transformation laws for T-dual coordinates ${}^\star\Theta^A$ and ${}^\star\bar{\Theta}^A$ have the same form as for initial ones Θ^A and $\bar{\Theta}^A$ we get the fermionic T-dual "generalized metric" ${}^\star\mathcal{F}_{AB}$ and T-dual currents, ${}^\star\bar{\mathcal{J}}_{+A}$ and ${}^\star\mathcal{J}_{-A}$

$${}^\star\mathcal{F}_{AB} = \mathcal{T}_A{}^C \mathcal{F}_{CD} \mathcal{T}^D{}_B , \quad {}^\star\bar{\mathcal{J}}_{+A} = \mathcal{T}_A{}^B \bar{\mathcal{J}}_{+B} , \quad {}^\star\mathcal{J}_{-A} = \mathcal{T}_A{}^B \mathcal{J}_{-B} . \tag{103}$$

The matrix $\mathcal{A}_{AB}$ transforms as

$${}^\star\mathcal{A}_{AB} = \mathcal{T}_A{}^C \mathcal{A}_{CD} \mathcal{T}^D{}_B = (\mathcal{A}^{-1})_{AB} . \tag{104}$$

From the first relation in (103) we obtain the form of the fermionic T-dual R-R background field

$${}^\star P_{\alpha\beta} = (P^{-1})_{\alpha\beta} , \tag{105}$$

while from the second and third equation we obtain the form of the fermionic T-dual NS-R background fields

$${}^\star\Psi_{\alpha\mu} = (P^{-1})_{\alpha\beta}\Psi^\beta_\mu , \quad {}^\star\bar{\Psi}_{\alpha\mu} = -\bar{\Psi}^\beta_\mu (P^{-1})_{\beta\alpha} . \tag{106}$$

The non singular matrix $\alpha^{\alpha\beta}$ transforms as

$$({}^\star\alpha)_{\alpha\beta} = (\alpha^{-1})_{\alpha\beta} . \tag{107}$$

As in the case of bosonic T-dualization, in the same way how we obtained the double action (51), we get the double action corresponding to the fermionic T-dual transformation law

$$\begin{aligned} & S_{double}(\Theta, \bar{\Theta}) = \qquad (108) \\ = \; & \frac{\kappa}{2} \int d^2\xi \Big[\partial_+ \bar{\Theta}^A \mathcal{F}_{AB} \partial_- \Theta^B + \bar{\mathcal{J}}_{+A} \partial_- \Theta^A + \partial_+ \bar{\Theta}^A \mathcal{J}_{-A} \\ - \; & \partial_- \bar{\Theta}^A \mathcal{A}_{AB} \partial_+ \Theta^B + L(x) \Big] , \end{aligned}$$

where $L(x)$ is arbitrary functional of the bosonic coordinates. In order to find fermionic T-dual metric and Kalb-Ramond field we suppose that term $L(x)$ has the form

$$L(x) = 2\partial_+ x^\mu \left(\Pi_{+\mu\nu} + {}^\star\Pi_{+\mu\nu}\right) \partial_- x^\nu \equiv \mathcal{L} + {}^\star\mathcal{L}\,. \tag{109}$$

This term should be invariant under T-dual transformation

$${}^\star\mathcal{L} = \mathcal{L} + \Delta\mathcal{L}\,. \tag{110}$$

Using the fact that two successive T-dualization are identity transformation, we obtain

$$\mathcal{L} = {}^\star\mathcal{L} + {}^\star\Delta\mathcal{L}\,. \tag{111}$$

Combining last two relations we get

$${}^\star\Delta\mathcal{L} = -\Delta\mathcal{L}\,. \tag{112}$$

If $\Delta\mathcal{L} = 2\partial_+ x^\mu \Delta_{\mu\nu} \partial_- x^\nu$, we obtain the condition for $\Delta_{\mu\nu}$

$${}^\star\Delta_{\mu\nu} = -\,\Delta_{\mu\nu}\,. \tag{113}$$

Using the relations (87) and (88) we realize that, up to multiplication constant, combination

$$\Delta_{\mu\nu} = \bar\Psi^\alpha_\mu (P^{-1})_{\alpha\beta} \Psi^\beta_\nu\,, \tag{114}$$

satisfies the condition (113). So, we conclude that

$${}^\star\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + c\bar\Psi^\alpha_\mu (P^{-1})_{\alpha\beta} \Psi^\beta_\nu\,, \tag{115}$$

where c is an arbitrary constant. For $c = \frac{1}{2}$ we obtain the relations (90).

5. Conclusions

In this article we showed that the new interpretation of T-dualization procedure in double space formalism offered in [26, 27] is also valid in the case of type II superstring theory - both for bosonic and fermionic T-dualization. We used the ghost free action of type II superstring theory in pure spinor formulation in the approximation of quadratic terms and constant background fields. One can obtain this action from action (1) under some set of assumptions.

We introduced the double space coordinate $Z^M = (x^\mu, y_\mu)$ adding to all bosonic initial coordinates, x^μ, the T-dual ones, y_μ. Then we rewrote the T-dual transformation laws (43) in terms of double space variables (45) introducing the generalized metric $\mathcal{H}_{MN}$ and the current $J_{\pm M}$. Further, we split initial coordinates x^μ in two parts: x^a are directions along which we made T-dulization and the rest ones x^i.

T-dualization is realized as permutation of the subsets x^a and y_a in the double space coordinate Z^M. Demanding that T-dual double space coordinates ${}_aZ^M = (\mathcal{T}^a)^M{}_N Z^N$ satisfy the T-dual transformation law of the same form as the initial coordinates Z^M we found the T-dual generalized metric ${}_a\mathcal{H}_{MN}$ and the T-dual current ${}_aJ_{\pm M}$. Consequently, we obtained the form of NS-NS and NS-R T-dual background fields in terms of the initial ones which are in full accordance with the results obtained by Buscher T-dualization procedure [6, 7].

In order to obtain T-dual R-R field strength $F^{\alpha\beta}$ we should make some additional assumptions. Supposing that term $L(\pi_\alpha, \bar{\pi}_\alpha)$ (51) is T-dual invariant and taking into account that two successive T-dualizations act as identity operator, we found the form of T-dual R-R field strength up to one arbitrary constant c. For $c = 4\kappa$ we get the T-dual R-R field strength ${}_aF^{\alpha\beta}$ as in Buscher procedure [6].

In the case of fermionic T-duality, using equations of motion with respect to the fermionic momenta π_α and $\bar{\pi}_\alpha$, we eliminated them from the action. Then we fixed local chiral gauge invariance using BRST approach.

Using the Buscher approach we performed fermionic T-duality procedure and obtained the form of the fermionic T-dual background fields. Analogously with the bosonic case we introduced double fermionic space doubling the initial coordinates θ^α and $\bar{\theta}^\alpha$ with their fermionic T-duals, ϑ_α and $\bar{\vartheta}_\alpha$. Double fermionic space is spanned by the coordinates $\Theta^A = (\theta^\alpha, \vartheta_\alpha)$ and $\bar{\Theta}^A = (\bar{\theta}^\alpha, \bar{\vartheta}_\alpha)$. Demanding that T-dual transformation laws for fermionic T-dual double coordinates, ${}^\star\Theta^A = \mathcal{T}^A{}_B\Theta^B$ and ${}^\star\bar{\Theta}^A = \mathcal{T}^A{}_B\bar{\Theta}^B$, are of the same form as those for Θ^A and $\bar{\Theta}^A$, we obtained the form of the fermionic T-dual NS-R and R-R background fields which are in full accordance with the results obtained by standard Buscher procedure. The expressions for T-dual metric ${}^\star G_{\mu\nu}$ and Kalb-Ramond field ${}^\star B_{\mu\nu}$ cannot be found from double space formalism because they do not appear in the T-dual transformation laws. These expressions, up to arbitrary constant, are obtained assuming that two successive T-dualization act as identity transformation.

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