The noncommutative $SO(2,3)_\star$ gravity model

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Abstract

In this review, noncommutative gravity is treated as a gauge theory of the noncommutative $SO(2,3)_\star$ group. We assume that the spacetime deformation is canonical. The Seiberg-Witten map is used to express noncommutative fields in terms of the corresponding commutative fields. In addition to pure gravity, we consider couplings to matter fields, in particular, fermion and $U(1)$ gauge field. The analysis can be extended to non Abelian gauge fields and scalar fields.

1. Introduction

Quantum Field Theory (QFT) and General Relativity (GR) are widely regarded as the two pillars of modern theoretical physics. Although these theories have been tested to an excellent degree of accuracy in their respective areas of applicability, occurrence of singularities in both of them strongly indicates that they are incomplete. GR, as a classical theory of gravitation, describes large-scale geometric structure of spacetime and its relation to the distribution of matter. On the other hand, QFT, standing on the principles of Quantum Mechanics and Special Relativity, provides us with the Standard Model of elementary particles which successfully utilizes the concept of gauge (local) symmetry to describe the fundamental

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particle interactions (electro-weak and strong). In spite of being fundamentally inconsistent with each other[1], both theories (QFT and GR) rely on the concepts of continuous spacetime and that of point particle. This is indeed appropriate, but, in the case of GR, only at large distances and only approximately. In order to establish the, still elusive, theory of "Quantum Gravity", i.e. the theory of the quantum structure of spacetime, at very short distances (very high energies) we must go beyond the usual concepts of QFT and GR to which we are accustomed. Variety of ways of treating the problem has been proposed so far, stemming from String Theory, Loop Quantum Gravity, Noncommutative (NC) Field Theory, etc.

Recently, a lot of attention has been devoted to NC gravity, as a special case of NC Field Theory that doesn't include matter fields, and many different approaches have been developed. In [1, 2, 3] a deformation of pure Einstein gravity via Seiberg-Witten (SW) map is proposed. Twist approach was explored in [4, 5, 6, 7]. Some other proposals are given in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. The connection to Supergravity was established in [21, 22]. Finally, the most relevant for our work is the approach developed by the members of our group [23, 24, 25, 26]. In this approach NC gravity is treated as a gauge theory of canonically deformed Anti de Sitter (AdS) group $SO(2,3)_\star$ (NC version of ordinary $SO(2,3)$ group).

A next natural step is to tackle the problem of introducing matter fields on NC spacetime and their interaction with NC gravity. Our theoretical model is the extension of the previously established $SO(2,3)_\star$ model of pure NC gravity [23, 24, 25, 26]. During the development of this model, the subject was treated from a different point of view by Aschieri and Castellani [27, 28, 29, 30]. Their model, based on the deformed $SO(1,3)$ symmetry, is significantly different from the one that we propose. On the formal side, in their $SO(1,3)_\star$ model, there is a problem of vierbein not being a well-defined gauge field. This is not an issue if we use $SO(2,3)$ gauge group. More importantly, our model leads to some very definite physical predictions and allows us to actually derive them in detail, while on the other hand no elaboration concerning any potential physical effects has been given by the upper mentioned authors. Also, we should emphasize that the differences between the two models revealed themselves already in the case of pure gravity, namely, the canonical NC deformation of Minkowski space actually is obtained in $SO(2,3)_\star$ model [23].

[1] Except in a certain narrow range of parameters where semi-classical discipline of QFT in curved spacetime background provides us with some reliable predictions, like the well-known Hawking radiation.

2. Introducing matter fields on NC spacetime

2.1. Deformation quantization

The construction of a NC Field Theory, i.e. Field Theory on NC spacetime, is based on the general method of deformation quantization via NC $\star$-products developed (mainly) by Flato, Sternheimer and Kontsevich [31, 32, 33]. One speaks of a *deformation* of an object/structure whenever there is a family of similar objects/structures for which we can parametrise their "distortion" from the original, "undeformed" one. In physics, this so called *deformation parameter* appears as some fundamental constant of nature that measures the deviation from the "classical" (undeformed) theory. When it is zero, the classical theory is restored. We want to deform the structure of continuous spacetime. This way of "quantizing" spacetime is essentially different from the standard QFT quantization procedure for matter fields. The spacetime coordinates (ordinary $3+1$) are proclaimed to be mutually *incompatible*. Analogously to the Heisenberg's uncertainty relations for a conjugate coordinate-momentum pair of a particle, there exist a non-zero lower bound for the product of uncertainties $\Delta x^\mu \Delta x^\nu$ for a pair of two different coordinates. In order to capture this "fuzziness" of spacetime, an abstract algebra of *NC coordinates* is introduced as a deformation of the ordinary structure. These NC coordinates, denoted by $\hat{x}^\mu$, satisfy some non-trivial commutation relations, and so, it is no longer the case that $\hat{x}^\mu\hat{x}^\nu = \hat{x}^\nu\hat{x}^\mu$. Abandoning this basic property of spacetime leads to various new physical effects that were not present in the theory based on classical spacetime. The simplest case of noncommutativity is the so called *canonical noncommutativity*, defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \ , \tag{2.1}$$

where $\theta^{\mu\nu}$ are components of a *constant* antisymmetric matrix. Other important choices include Lie algebra-like deformation and κ-deformation.

Instead of deforming abstract algebra of coordinates one can take alternative, but equivalent, approach in which noncommutativity appears in the form of NC products of functions (NC fields) of *commutative* coordinates. These products are called *star products* ($\star$-products). Specifically, to establish *canonical noncommutativity*, we use NC Moyal-Weyl $\star$-product,

$$(\hat{f} \star \hat{g})(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}} f(x)g(y)|_{y\to x} \ . \tag{2.2}$$

The first term in the expansion of the exponential is the ordinary commutative point-wise multiplication of functions. Using this definition one an easily check that

$$[x^\mu \overset{\star}{,} x^\nu] = i\theta^{\mu\nu} \ , \tag{2.3}$$

which means that the Moyal-Weyl $\star$-product (2.2) provides a suitable representation of the canonical noncommutativity. The quantity $\theta^{\mu\nu}$ is a "small" *deformation parameter* that has dimensions of $(length)^2$. It is a fundamental constant like the Planck length or the speed of light.

2.2. Enveloping algebra approach and the Seiberg-Witten map

In classical (undeformed) gauge field theories, generators T_A ($A = 1, 2, ..., N$) of a gauge group (generally non-Abelian) satisfy some N-dimensional Lie algebra commutation relations:

$$[T_A, T_B] = f^C{}_{AB} T_C \ , \tag{2.4}$$

with structure constants $f^C{}_{AB}$. Variation of the matter field ψ under infinitesimal gauge transformation is given by:

$$\delta_\alpha \psi = i\alpha\psi \ , \tag{2.5}$$

where infinitesimal gauge parameter $\alpha(x) = \alpha^A(x)T_A$ belongs to the Lie algebra of the gauge group and depends on the spacetime coordinates. These transformations close in the algebra:

$$[\delta_\alpha, \delta_\beta] = \delta_{-i[\alpha,\beta]} \ . \tag{2.6}$$

Covariant derivative of the ψ field is

$$D_\mu\psi = \partial_\mu - i\omega_\mu\psi \ , \tag{2.7}$$

where $\omega_\mu(x) = \omega_\mu^A(x)T_A$ is the Lie algebra-valued gauge potential.

Analogously, a variation of a NC matter field $\widehat{\psi}$ under deformed infinitesimal gauge transformation with NC gauge parameter $\widehat{\Lambda}(x)$ is defined as:

$$\delta^\star_\Lambda \widehat{\psi} = i\widehat{\Lambda} \star \widehat{\psi} \ . \tag{2.8}$$

However, there is a problem concerning the closure condition for NC gauge transformations. If the parameter $\widehat{\Lambda}$ is supposed to be Lie algebra-valued, $\widehat{\Lambda}(x) = \widehat{\Lambda}^A(x)T_A$, it follows that

$$\begin{aligned}
[\delta^\star_{\Lambda_1} \stackrel{\star}{,} \delta^\star_{\Lambda_2}]\widehat{\psi} &= (\widehat{\Lambda}_1 \star \widehat{\Lambda}_2 - \widehat{\Lambda}_2 \star \widehat{\Lambda}_1) \star \widehat{\psi} \\
&= \frac{1}{2}\Big([\widehat{\Lambda}_1^A \stackrel{\star}{,} \widehat{\Lambda}_2^B]\{T_A, T_B\} + \{\widehat{\Lambda}_1^A \stackrel{\star}{,} \widehat{\Lambda}_2^B\}[T_A, T_B]\Big) \star \widehat{\psi}
\end{aligned} \tag{2.9}$$

The commutator of two NC gauge transformations does not generally close in the Lie algebra of gauge group because the anti-commutator $\{T_A, T_B\}$ does not in general belong to this algebra. To overcome this difficulty, we apply the *enveloping algebra approach.* The enveloping algebra is "large enough" to ensure the closure property of NC gauge transformations if NC gauge parameter $\widehat{\Lambda}$ takes values in it. The covariant derivative of NC field $\widehat{\psi}$ is defined by:

$$D_\mu\widehat{\psi} = \partial_\mu\widehat{\psi} - i\widehat{\omega}_\mu \star \widehat{\psi} \ . \tag{2.10}$$

Here we introduced NC gauge potential $\widehat{\omega}_\mu$ that also belongs to the enveloping algebra and can be represented in its basis. But, enveloping algebra has an infinite basis and it seems that by invoking it we are actually introducing infinite number of new degrees of freedom (new fields) in the NC-deformed theory. The solution to the problem is provided by the Seiberg-Witten map [34, 35]. It relies on the fact that NC quantities can be represented as a perturbation series in powers of the deformation parameter $\theta^{\alpha\beta}$ with expansion coefficients built out of the commutative fields, e.g. NC spinor $\widehat{\psi}$ field can be represented as:

$$\widehat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2) \ . \tag{2.11}$$

where ω_α is the ordinary gauge potential. It is clear that at the zeroth perturbative order NC fields consistently reduce to their undeformed counterparts.

2.3. Commutative model and its NC deformation

The first step in our method is to construct a well-defined "classical" (undeformed) *commutative model* of an action that will subsequently be *deformed* by substituting ordinary commutative point-wise multiplication with NC Moyal-Weyl $\star$-product and replacing ordinary commutative fields by their NC counterparts. Classical action is built out of the commutative fields on *Minkowski space* that transform either in the fundamental or the adjoint representation of the gauge symmetry group. For pure gravity this will be $SO(2,3)$ "gravitational" gauge symmetry but if we want to include interacting Dirac fermions in the theory we must upgrade it to $SO(2,3) \otimes U(1)$. The SW map than ensures that NC-deformed action possesses deformed gauge symmetry, i.e. $SO(2,3)_\star \otimes U(1)_\star$. Gravity emerges only after suitable *gauge fixing*, i.e. gauge symmetry breaking, and there is no need for introducing metric tensor a priori. The commutative action must be consistent with the action for *classical electrodynamics in curved spacetime* with the usual $SO(1,3) \otimes U(1)$ gauge symmetry. Therefore we must break the $SO(2,3)$ gauge symmetry down to the local Lorentz $SO(1,3)$ symmetry. We accomplish this by introducing an *auxiliary field* ϕ and fixing its value. Assuming that deformation parameter is "small", we *perturbatively expend* the NC action in powers of the deformation parameter via SW map. In this way we ensure the invariance of the NC expansion under original udeformed gauge transformations order-by-order. The *first order term* in the NC expansion contains the "strongest" NC effects and for that reason a model that predicts non vanishing first order NC correction after the symmetry breaking would be preferred.

2.4. $SO(2,3)_\star$ model of NC gravity

The $SO(2,3)_\star$ model of NC gravity is established in [23, 24, 25, 26]. In this approach, NC gravity is treated as as a *gauge theory* of *canonically* *deformed* Anti de Sitter (AdS) gauge group $SO(2,3)_\star$. The generators of $SO(2,3)$ group are denoted by M_{AB} (group indices $A, B, ...$ taking values: $0, 1, 2, 3, 5$) and they satisfy the AdS algebra:

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) \ , \quad (2.12)$$

where η_{AB} is the $5D$ flat metric with signature $(+,-,-,-,+)$. By introduce *momenta generators* as $P_a = \frac{1}{l}M_{a5}$, where l is a constant length scale, we can recast the AdS algebra (2.12) in the following form:

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \ , &(2.13)\\ [M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b) \ , &(2.14)\\ [P_a, P_b] &= -\frac{i}{2l^2}M_{ab} \ . &(2.15) \end{aligned}$$

In the limit $l \to \infty$ the AdS algebra reduces to the Poincare algebra. This is known as the Wigner-Inonu contraction. A realisation of (2.12) algebra can be obtained from $5D$ gamma matrices Γ^A that satisfy Clifford algebra: $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$; the generators are given by $M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B]$. One choice of $5D$ gamma matrices is $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a are the usual $4D$ gamma matrices. The local Lorentz indices $a, b, ...$ take values: $0, 1, 2, 3$. In this particular representation, $SO(2,3)$ generators are: $M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab}$ and $M_{5a} = \frac{1}{2}\gamma_a$. The gauge potential ω_μ of $SO(2,3)$ gauge group gives us the spin-connection ${\omega_\mu}^{ab}$ and the vierbein e^a_μ field:

$$\omega_\mu = \frac{1}{2}{\omega_\mu}^{AB}M_{AB} = \frac{1}{4}{\omega_\mu}^{ab}\sigma_{ab} - \frac{1}{2l}e^a_\mu\gamma_a \ . \quad (2.16)$$

Note that in this framework the vierbein field e^a_μ is treated as an additional gauge field, standing on equal footing with the spin-connection (which is the gravitational gauge field for the Lorentz group $SO(1,3)$). This is an important advantage of the theory with $SO(2,3)$ gauge symmetry. Vierbein is related to the metric tensor by $\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu}$ and $e = \det(e^a_\mu) = \sqrt{-g}$.

The "gravitational" field strength associated with the gauge potential ω_μ is

$$F_{\mu\nu} = 2\partial_{(\mu}\omega_{\nu)} - i[\omega_\mu, \omega_\nu] = \Big({R_{\mu\nu}}^{ab} - \frac{2}{l^2}e^a_{(\mu}e^b_{\nu)}\Big)\frac{\sigma_{ab}}{4} - \frac{1}{l}{T_{\mu\nu}}^a\frac{\gamma_a}{2} \ , \quad (2.17)$$

where ${R_{\mu\nu}}^{ab}$ is the *curvature tensor* and ${T_{\mu\nu}}^a$ the *torsion*:

$$\begin{aligned} {R_{\mu\nu}}^{ab} &= \partial_\mu{\omega_\nu}^{ab} - \partial_\nu{\omega_\mu}^{ab} + {\omega_\mu}^{ac}{\omega_\nu}^{cb} - {\omega_\mu}^{bc}{\omega_\nu}^{ca} \ , &(2.18)\\ {T_{\mu\nu}}^a &= \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu \ . &(2.19) \end{aligned}$$

A necessary step in obtaining GR from $SO(2,3)$ model is the *gauge fixing*, i.e. symmetry breaking from $SO(2,3)$ down to $SO(1,3)$. For that reason one usually introduces an *auxiliary field* $\phi = \phi^A\Gamma_A$ as in [36, 37]. We break the symmetry by fixing the value of the auxiliary field, in particular, by setting $\phi^a = 0$ and $\phi^5 = l$. This field is a spacetime scalar and an internal-space vector and satisfies the constraint: $\phi^A\phi_A = l^2$. It transforms in the adjoint representation of $SO(2,3)$ and its covariant derivative is given by

$$D_\mu\phi = \partial_\mu\phi - i[\omega_\mu, \phi] \ . \tag{2.20}$$

After the gauge fixing, the components of $D_\mu\phi$ reduce to $(D_\mu\phi)^a = e_\mu^a$ and $(D_\mu\phi)^5 = 0$. In this way the vierbein field emerges from the auxiliary field ϕ after the symmetry breaking.

In the papers of Stelle, West and Wilczek [36, 37] a commutative action for *pure gravity* with $SO(2,3)$ gauge symmetry was constructed. Also, in the papers of Chamseddine and Mukhanov [38, 39], GR is formulated by gauging $SO(1,4)$ or, more suitable for SUGRA, $SO(2,3)$ group. Building on their work, in [24] the $SO(2,3)$ model of pure gravity action and its NC deformation were analysed. The commutative action consists of three parts:

$$\begin{aligned} S_1 &= \frac{ilc_1}{64\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\phi \ , && (2.21)\\ S_2 &= \frac{c_2}{128\pi G_N l}\mathrm{Tr}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}D_\rho\phi D_\sigma\phi\phi + h.c. \ , && (2.22)\\ S_3 &= -\frac{ic_3}{128\pi G_N l}\mathrm{Tr}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi \ , && (2.23)\end{aligned}$$

After the gauge fixing we obtain:

$$\begin{aligned} S &= -\frac{1}{16\pi G_N}\int d^4x\left(\frac{c_1 l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} + \sqrt{-g}\Big((c_1+c_2)R\right.\\ &\left.- \frac{6}{l^2}(c_1 + 2c_2 + 2c_3)\Big)\right) \ . \end{aligned} \tag{2.24}$$

For the sake of generality, three a priori undetermined dimensionless constants are introduced. These can be fixed by some consistency conditions. The first part is the topological Gauss-Bonnet term which doesn't effect the equations of motion, and so, we can set $c_1 = 0$. The Einstein-Hilbert term requires $c_1 + c_2 = 1$, while the absence of the cosmological constant is provided with $c_1 + 2c_2 + 2c_3 = 0$.

To construct a NC gravity model, we start from a general action of the form

$$\widehat{S} = c_1\widehat{S}_1 + c_2\widehat{S}_2 + c_3\widehat{S}_3 \, , \tag{2.25}$$

with

$$
\begin{aligned}
\widehat{S}_1 &= \frac{il}{64\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\star\hat{\phi}\,,\\
\widehat{S}_2 &= \frac{1}{64\pi G_N l}\mathrm{Tr}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\hat{\phi}\star\hat{F}_{\mu\nu}\star\hat{D}_\rho\hat{\phi}\star\hat{D}_\sigma\hat{\phi} + c.c.\,,\\
\widehat{S}_3 &= -\frac{i}{128\pi G_N l}\mathrm{Tr}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}D_\mu\hat{\phi}\star D_\nu\hat{\phi}\star\hat{D}_\rho\hat{\phi}\star\hat{D}_\sigma\hat{\phi}\star\hat{\phi}\,.
\end{aligned}
$$

The action is formulated in the 4-dimensional Minkowski space as an ordinary NC gauge theory. It is invariant under the NC $SO(2,3)_\star$ gauge transformations and the SW map ensures that after the perturbative expansion in powers of $\theta^{\alpha\beta}$, it possesses the ordinary commutative $SO(2,3)$ gauge symmetry order-by-order in $\theta^{\alpha\beta}$. It was found that the first order NC correction to the commutative action equals zero. This result was also obtained in [40]. The first non-vanishing correction is *quadratic* in the NC parameter; it is long and difficult to calculate. However, different limits of the second order correction can be analyzed. We are interested in the low energy expansion: we keep only the terms of the zeroth, the first and the second order in the derivatives of the vierbeins (linear in $R_{\alpha\beta\gamma\delta}$, quadratic in $T_{\alpha\beta}{}^a$). Then, the expanded action is given by

$$
\begin{aligned}
&S_{NC} = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\, e\left(R-\frac{6}{l^2}(1+c_2+2c_3)\right)\\
&+\frac{1}{128\pi G_N l^4}\int \mathrm{d}^4x\, e\,\theta^{\alpha\beta}\theta^{\gamma\delta}\Big((-2+12c_2+38c_3)R_{\alpha\beta\gamma\delta}\\
&+(4-18c_2-44c_3)R_{\alpha\gamma\beta\delta}-(6+22c_2+36c_3)g_{\beta\delta}R_{\alpha\mu\gamma}{}^{\mu}+\frac{6+28c_2+56c_3}{l^2}g_{\alpha\gamma}g_{\beta\delta}\\
&+(5-\frac{9}{2}c_2-7c_3)T^a_{\alpha\beta}T_{\gamma\delta a}+(-10+\frac{9}{2}c_2+14c_3)T^a_{\alpha\gamma}T_{\beta\delta a}+(3-3c_2-2c_3)T_{\alpha\beta\gamma}T_{\delta\mu}{}^{\mu}\\
&+(1+2c_2)T_{\alpha\beta\rho}T^{\rho}{}_{\gamma\delta}+8T_{\alpha\gamma\delta}T_{\beta\mu}{}^{\mu}-(2c_2+4c_3)T_{\alpha\gamma\rho}T^{\rho}{}_{\delta\beta}\\
&+(2c_2+4c_3)g_{\beta\delta}T_{\gamma\sigma}{}^{\sigma}T_{\alpha\rho}{}^{\rho}-(2c_2+4c_3)T_{\alpha\rho\sigma}T_{\gamma}{}^{\sigma\rho}g_{\beta\delta}+(-2+4c_2+18c_3)T_{\alpha\beta\gamma}e^{\rho}_a\nabla_\delta e^a_\rho\\
&+(6-8c_2-8c_3)T_{\alpha\gamma\beta}e^{\rho}_a\nabla_\delta e^a_\rho+(2+4c_2+12c_3)T_{\alpha\gamma}{}^{\mu}e^a_\beta\nabla_\delta e^a_\mu-T_{\alpha\beta}{}^{\mu}e^a_\delta\nabla_\gamma e^a_\mu\\
&+(-6-8c_2-16c_3)T_{\delta\rho\beta}e^{\rho}_a\nabla_\alpha e^a_\gamma-(2c_2+4c_3)g_{\alpha\gamma}T_{\mu\beta}{}^{\mu}e^{\rho}_a\nabla_\delta e^a_\rho-(2c_2+4c_3)g_{\beta\delta}T_{\alpha\rho}{}^{\sigma}e^{\rho}_a\nabla_\gamma e^a_\sigma\\
&-(4+16c_2+32c_3)e^{\mu}_a e_{b\beta}\nabla_\gamma e^a_\alpha\nabla_\delta e^b_\mu+(4+12c_2+32c_3)e_{\delta a}e^{\mu}_b\nabla_\alpha e^a_\gamma\nabla_\beta e^b_\mu\\
&-(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\nu}_b\nabla_\gamma e^a_\mu\nabla_\alpha e^b_\nu+(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\rho}_c\nabla_\alpha e^a_\rho\nabla_\gamma e^c_\mu\Big)\,.
\end{aligned}
\tag{2.26}
$$

Equations of motion are obtained by varying with respect to the vierbeins and the spin connection independently:

$$
\begin{aligned}
\delta e^a_\mu: \quad & R_{\alpha\gamma}{}^{cd}e^{\gamma}_d e^{\alpha}_a e^{\mu}_c-\frac{1}{2}e^{\mu}_a R+\frac{3}{l^2}(1+c_2+2c_3)e^{\mu}_a=\tau_a{}^{\mu}\\
&= -\frac{8\pi G_N}{e}\frac{\delta S^{(2)}_{NC}}{\delta e^a_\mu}\,,
\end{aligned}
\tag{2.27}
$$

$$
\delta\omega_\mu{}^{ab}: \quad T_{ac}{}^{c}e^{\mu}_b-T_{bc}{}^{c}e^{\mu}_a-T_{ab}{}^{\mu}=S_{ab}{}^{\mu}=-\frac{16\pi G_N}{e}\frac{\delta S^{(2)}_{NC}}{\delta\omega_\mu{}^{ab}}\,.
\tag{2.28}
$$

The effective energy-momentum tensor ${\tau_a}^\mu$ and the effective spin-tensor ${S_{ab}}^\mu$ depend on $\theta^{\alpha\beta}$ and we can conclude that *noncommutativity is a source of curvature and torsion.* Using equation (2.27) it can be shown [23] that a NC correction to Minkowski metric is of the form:

$$\begin{aligned} g_{00} &= 1-\frac{11}{2l^6}{\theta^0}_m{\theta^0}_n x^m x^n - \frac{11}{8l^6}\theta^2 r^2 = 1 - R_{0m0n}x^m x^n \,, \\ g_{0i} &= -\frac{11}{3l^6}{\theta^0}_m{\theta^i}_n x^m x^n = -\frac{2}{3}R_{0min}x^m x^n \,, \\ g_{ij} &= -\delta_{ij} - \frac{11}{6l^6}{\theta^i}_m{\theta^j}_n x^m x^n + \frac{11}{24l^6}\delta^{ij}\theta^2 r^2 - \frac{11}{24l^6}\theta^2 x^i x^j \\ &= -\delta_{ij} - \frac{1}{3}R_{imjn}x^m x^n \,, \end{aligned} \tag{2.29}$$

where $R_{\mu\nu\rho\sigma}$ are the components of the Reimann tensor for this solution.

Thus, in $SO(2,3)_\star$ model, there actually exists a *canonical NC deformation of Minkowski space.* This result suggests that the coordinates x^μ we started with are actually *Fermi normal coordinates.* These are inertial coordinates of a local observer moving along a geodesic and can be constructed in a small neighbourhood along the geodesic (inside a small cylinder surrounding the geodesic) [41, 42, 43]. The breaking of diffeomorphism symmetry due to canonical noncommutivity we understand as a consequence of working in a preferred reference system given by the Fermi normal coordinates. A local observer moving along the geodesic measures $\theta^{\alpha\beta}$ to be constant. In any other reference frame this will not be the case.

3. Commutative actions for matter fields and their NC deformation

First, we will consider *non-interacting fermions* in canonically deformed spacetime. In our recant paper entitled *Dirac field and gravity in NC $SO(2,3)_\star$ model* [44] we have proposed the following kinetic-type action for the Dirac spinor field, invariant under local $SO(2,3)$ transformations:

$$S_{\psi,kin} = \frac{i}{12}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi - D_\sigma\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\psi\Big] \,. \tag{3.30}$$

After the symmetry braking it becomes:

$$S_{\psi,kin} = \frac{i}{2}\int d^4x\, e\, \left[\bar\psi\gamma^\sigma\nabla_\sigma\psi - \nabla_\sigma\bar\psi\gamma^\sigma\psi\right] - \frac{2}{l}\int d^4x\, e\, \bar\psi\psi \,, \tag{3.31}$$

which is exactly the Dirac action in curved spacetime for spinors of mass $2/l$. To obtain fermions with arbitrary mass m, not just $2/l$, we have to include the following additional "mass terms" (terms of the type $\bar\psi...\psi$) in the action:

$$
\begin{aligned}
S^{(1)}_{\psi,m} &= \frac{i}{2}c_1\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi+\bar\psi\phi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big]\ ,\\
S^{(2)}_{\psi,m} &= \frac{i}{2}c_2\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi+\bar\psi D_\mu\phi\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big]\ ,\\
S^{(3)}_{\psi,m} &= i\ c_3\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \ \bar\psi D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi\ .
\end{aligned}
\tag{3.32}
$$

If we demand that the a priori undetermined dimensionless coefficients c_1, c_2, and c_3 satisfy the constraint: $c_2-c_1-c_3=-\frac{1}{24}$, after the symmetry breaking, the sum of the three terms in (3.32) becomes:

$$
S_{\psi,m}=-\left(m-\frac{2}{l}\right)\int d^4x\ e\ \bar\psi\psi\ , \tag{3.33}
$$

and the total action, $S_\psi=S_{\psi,kin}+S_{\psi,m}$, is exactly equal to the Dirac action for spinors of mass m in curved spacetime,

$$
S_\psi=\frac{i}{2}\int d^4x\ e\ \left[\bar\psi\gamma^\sigma\nabla_\sigma\psi-\nabla_\sigma\bar\psi\gamma^\sigma\psi\right]-m\int d^4x\ e\ \bar\psi\psi\ . \tag{3.34}
$$

NC-deformed version of this spinorial action (denoted by a "hat" symbol) is obtained by replacing the ordinary field multiplication with NC $\star$-product of NC-deformed fields; for example, kinetic term (3.30) becomes:

$$
\begin{aligned}
\widehat{S}_{\psi,kin}=\frac{i}{12}\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \Big[&\widehat{\bar\psi}\star(D_\mu\widehat{\phi})\star(D_\nu\widehat{\phi})\star(D_\rho\widehat{\phi})\star(D_\sigma\widehat{\psi})\\
&-(D_\sigma\widehat{\bar\psi})\star(D_\mu\widehat{\phi})\star(D_\nu\widehat{\phi})\star(D_\rho\widehat{\phi})\star\widehat{\psi}\Big]\ .
\end{aligned}
\tag{3.35}
$$

This action is hermitian (up to a surface term that vanishes) and it is endowed with deformed $SO(2,3)_\star$ gauge symmetry. It turns out that the leading term in the NC expansion does not vanish after the symmetry breaking and so we obtain linear NC correction to the classical Dirac action in curved spacetime. The calculation is long and tedious and we will not present the details here. Schematically, the spinorial piece is given by:

$$
\widehat{S}^{(1)}_\psi=\theta^{\alpha\beta}\int d^4x\ e\ \bar\psi\Big(\mathcal{A}_{\alpha\beta}{}^{\rho\sigma}\nabla_\rho\nabla_\sigma+\mathcal{B}_{\alpha\beta}{}^{\sigma}\nabla_\sigma+\mathcal{C}_\alpha\nabla_\beta+\mathcal{D}_{\alpha\beta}\Big)\psi\ . \tag{3.36}
$$

Objects $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are complicated functions of geometric quantities, e.g. there appear new interactions like: $\bar\psi\sigma_\alpha{}^\sigma\nabla_\beta\nabla_\sigma\psi$, $\bar\psi R_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\nabla_\sigma\psi$, $\bar\psi T_{\alpha\beta}{}^{\sigma}\nabla_\sigma\psi$, $\bar\psi\sigma_{\alpha\beta}\psi$, etc. More importantly, we want to emphasize the fact that this θ-linear NC correction pertains even in the limit of *flat spacetime.* This enables us to derive some tangible phenomenological consequences of our model that could potentially be tested experimentally in a not to far future.

3.1. Free electron in flat NC spacetime

One peculiar property of our model is that noncommutativity effects pertain even in flat spacetime. The NC-deformed Dirac equation for an electron in Minkowski space can be derived by varying the action (3.36) with respect to $\bar{\psi}$. In particular, assuming for simplicity that $\theta^{12} = -\theta^{21} = \theta \neq 0$ and all others equal to zero, we obtain:

$$\Big[i\partial\!\!\!/ - m - \frac{\theta}{2l}(\sigma_1{}^\sigma\partial_2\partial_\sigma - \sigma_2{}^\sigma\partial_1\partial_\sigma) + \frac{7i\theta}{12l^2}(\gamma_0\gamma_5\partial_3 - \gamma_3\gamma_5\partial_0) \\ -\theta\left(\frac{m}{2l^2}+\frac{1}{3l^3}\right)\sigma_{12}\Big]\psi = 0\ . \qquad (3.37)$$

Non trivial solutions of this homogeneous matrix equation exist, if and only if, the determinant of the matrix that acts on ψ equals zero. This condition will give us the *dispersion relation* for an electron. Specifically, for an electron moving along the z-direction, i.e. in the direction orthogonal to the noncommutative x, y-plane there are four different solutions for the energy function:

$$\begin{aligned} E_{1,2}(\mathbf{p}) &= E_\mathbf{p} \mp \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_\mathbf{p}} + \mathcal{O}(\theta^2)\ , \\ E_{3,4}(\mathbf{p}) &= -E_\mathbf{p} \pm \left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right]\frac{\theta}{E_\mathbf{p}} + \mathcal{O}(\theta^2)\ , \end{aligned} \qquad (3.38)$$

with $E_\mathbf{p} = \sqrt{m^2 + p_z^2}$. This is reminiscent of the well known *Zeeman effect*. The deformation parameter θ plays the role of a constant background magnetic field that causes the splitting of electron energy levels.

Working in the rest frame ($\mathbf{p} = 0$), from (3.38) we conclude that electron's mass gets modified due to noncommutativity of the background spacetime and the correction is *linear* in the deformation parameter:

$$\begin{aligned} E_{1,2}(0) &= m \mp \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta + \mathcal{O}(\theta^2)\ , \\ E_{3,4}(0) &= -m \pm \left[\frac{m}{12l^2} - \frac{1}{3l^3}\right]\theta + \mathcal{O}(\theta^2)\ . \end{aligned} \qquad (3.39)$$

Another important prediction of $SO(2,3)_\star$ model is that NC energy levels are *helicity-dependent*. The electron states (presented in [44]) with different helicity have different energies. This means that NC space effectively acts as a *birefringent medium* for electrons propagating in it.

Such NC modification of the *non-interacting theory* could not have be achieved by directly introducing noncommutativity into the free Dirac action, i.e. by minimal substitution,

$$\widehat{S} = \int d^4x \ \widehat{\bar{\psi}} \star (i\gamma^\mu \partial_\mu - m)\widehat{\psi} \ , \tag{3.40}$$

because generally $\int d^4x \ \widehat{f} \star \widehat{g} = \int d^4x \ fg$, and so, non-interacting theories are not modified due to noncommutativity since they include only kinetic term which is quadratic.

3.2. Interacting Dirac fermions; U(1) gauge field

In order to have interacting Dirac fermions we need to incorporate $U(1)$ gauge field into our framework. Accordingly, the gauge group must be upgraded from $SO(2,3)$ to $SO(2,3) \otimes U(1)$. In the paper entitled *Noncommutative Electrodynamics from $SO(2,3)_\star$ model of Noncommutative Gravity* [45] (under revision), we propose the following action for the $U(1)$ gauge field:

$$S_A = -\frac{1}{16l} \, Tr \int d^4x \ \varepsilon^{\mu\nu\rho\sigma} \Big\{ f \mathbb{F}_{\mu\nu} + \frac{i}{3!} f^2 D_\mu \phi D_\nu \phi \Big\} D_\rho \phi D_\sigma \phi \phi + h.c. \tag{3.41}$$

It includes an additional auxiliary field $f = \frac{1}{2} f^{AB} M_{AB}$. Like ϕ, this field transforms in the adjoint representation of $SO(2,3)$ and it is invariant under $U(1)$ (i.e. not charged). Its role is to produce the canonical kinetic term for $U(1)$ gauge field in curved spacetime since we cannot define the Hodge dual operation without prior knowledge of the metric tensor.

After the gauge fixing, the purely gravitational part of the action (3.41) vanishes and we obtain:

$$S_A = \frac{1}{2} \int d^4x \ e \ f^{ab} e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu} + \frac{1}{4} \int d^4x \ e \ (f^{ab} f_{ab} + 2 f^{a5} f_a{}^5) \ . \tag{3.42}$$

We use equations of motion for the components of the auxiliary field f,

$$f_{a5} = 0 \ , \quad f_{ab} = -e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu} \ . \tag{3.43}$$

to eliminate the auxiliary field in the action (3.42) and this leaves us with the canonical kinetic term for pure $U(1)$ gauge field in curved spacetime:

$$S_A = -\frac{1}{4} \int d^4x \ e \ \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \ . \tag{3.44}$$

3.3. Electron in background magnetic field

With the $U(1)$ gauge field include, we can investigate the behaviour of an electron in background electromagnetic field, see [45] for detail. The action for NC electrodynamics in Minkowski space up to the first order in $\theta^{\alpha\beta}$ is given by:

$$\begin{aligned}
\widehat{S}_{flat} &= \int d^4x\, \bar{\psi}(i\not{\mathcal{D}} - m)\psi - \frac{1}{4}\int d^4x\, \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \\
&\quad -\theta^{\alpha\beta}\int d^4x\, \left[\frac{1}{8}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\mathcal{F}^{\mu\nu}\right] \\
&\quad +\theta^{\alpha\beta}\int d^4x\, \bar{\psi}\Bigg[-\frac{1}{2l}\sigma_\alpha{}^\sigma\mathcal{D}_\beta\mathcal{D}_\sigma + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\mathcal{D}_\sigma - \left(\frac{m}{4l^2} + \frac{1}{6l^3}\right)\sigma_{\alpha\beta} \\
&\quad +\frac{3i}{4}\mathcal{F}_{\alpha\beta}\not{\mathcal{D}} - \frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^\mu\mathcal{D}_\beta - \left(\frac{3m}{4} - \frac{1}{4l}\right)\mathcal{F}_{\alpha\beta}\Bigg]\psi\ . \qquad (3.45)
\end{aligned}$$

where we introduced the flat spacetime covariant derivative $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ (for electron $q = -1$). We notice immediately that this action is different from the actions for NC electrodynamics already present in the literature [46, 47, 48]. The new interaction terms, specific to the $SO(2,3)_\star$ model, appear as residual from the gravitational interaction and they lead to some non trivial physical consequences, such as, for instance, a deformation of electron's dispersion relation.

By varying NC action (3.45) with respect to $\bar{\psi}$ we obtain NC-deformed Dirac equation for electron coupled to some background electromagnetic field A_μ:

$$\left(i\not{\partial} - m + \not{A} + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\right)\psi = 0\ . \qquad (3.46)$$

In [45] we investigate a special case of an electron propagating in constant background magnetic field $\mathbf{B} = B\mathbf{e}_z$ in order to see how noncommutativity deforms its energy levels.

Classical (undeformed) energy levels for a *relativistic electron* are given by:

$$E^{(0)}_{n,s} = \sqrt{p_z^2 + m^2 + (2n+s+1)B}\ . \qquad (3.47)$$

where $s = \pm 1$ is the σ_z eigenvalue for the spin-state of an electron. We are looking for linear NC correction $E^{(1)}_{n,s} \sim \theta$ of the energy levels (3.47). If we assume that only two spatial coordinates are mutually incompatible, e.g. $[x^1, x^2] = i\theta^{12}$, than we have $\theta^{12} = -\theta^{21} =: \theta \neq 0$ and all other components of $\theta^{\alpha\beta}$ equal to zero. In this case we obtain:

$$\begin{aligned}
E^{(1)}_{n,s} = -\frac{\theta s}{E^{(0)}_{n,s}}\left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right] - \frac{\theta B s}{E^{(0)}_{n,s}(E^{(0)}_{n,s}+m)}\left[\frac{m}{12l^2} - \frac{1}{3l^3}\right](2n+s+1) \\
+\frac{\theta B^2}{2E^{(0)}_{n,s}}(2n+s+1)\ . \qquad (3.48)
\end{aligned}$$

Non-relativistic limit of NC energy levels is obtained by expanding undeformed energy function $E^{(0)}_{n,s}$ assuming that p_z^2, $B \ll m^2$; for a *non-relativistic electron* constrained to move in NC x, y-plane ($p_z = 0$) expending (3.48) we obtain:

$$E_{n,s} = \left[m - s\theta \left(\frac{m}{12l^2} - \frac{1}{3l^3} \right) \right] + \frac{2n+s+1}{2m} B_{eff} - \frac{(2n+s+1)^2}{8m^3} B_{eff}^2 \; , \tag{3.49}$$

where we introduced $B_{eff} = (B + \theta B^2)$ as an *effective magnetic field*.

If we compare the expression for NC-deformed energy levels (3.49) with the one for undeformed energy levels, we conclude that the only effect of constant spacetime noncommutaivity is to modify the mass of an electron and the value of the background magnetic field. This interpretation is in accord with String Theory. In the famous paper of Seiberg and Witten [35] it is argued that coordinate functions of the endpoints of an open string constrained to a D-brane in the presence of a constant Neveu-Schwarz B-field (equivalent to a constant magnetic field on the brane-world) satisfy constant noncommutativity algebra. The implication is that a relativistic field theory on NC spacetime can be interpreted as a low energy limit of the theory of open strings.

From energy function (3.49) we can derive NC-deformed *magnetic moment of an electron* in the state (n, s) for weak magnetic field:

$$\mu_{n,s} = -\frac{\partial E_{n,s}}{\partial B} = -\mu_B (2n + s + 1)(1 + \theta B), \tag{3.50}$$

where, if we restore the units, $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton. We recognise $-(2n + 1)\mu_B$ as the *diamagnetic moment* of an electron and $-s\mu_B$ as the spin magnetic moment. The θB-term is another potentially observable phenomenological prediction of our model. It is a linear NC correction to the electron's dipole moment.

4. Conclusion

In this article we discussed coupling of matter fields with gravity in the framework of NC $SO(2,3)_\star$ gauge theory of gravity. Taking the enveloping algebra approach, along with the Seiberg-Witten map, we constructed the perturbative actions and derived the *deformed* equations of motion for the Dirac field coupled to $U(1)$ gauge field and gravity. In this way we have established the NC Electrodynamics in curved spacetime induced by NC $SO(2,3)_\star$ gravity. The fact that NC effects pertain even in flat spacetime limit of this model enables one to study behaviour of an electron in a background electromagnetic field. Corrections to the relativistic Landau levels of an electron in a constant magnetic field are given by (3.48) and their non-relativistic limit is (3.49). The NC corrections are different for different Landau levels. It is well known that the physics of the Lowest Landau Level (LLL) is closely related to the physics of Quantum Hall Effect (QHE). Using the obtained results, we plan to investigate NC corrections to the QHE. In this way, together with the induced NC magnetic moment (3.50) and the NC-induced magnetization in materials we hope to obtain some constraints on noncommutaivity parameter from condensed matter experiments.

Starting from (3.45) one can check renormalizability of the model. It is known that, the so-called Minimal NC Electrodynamics, a theory obtained by directly introducing NC Moyal-Weyl $\star$-product in the classical Dirac action for fermions coupled with $U(1)$ gauge field in Minkowski space,

$$\widehat{S} = \int d^4x\, \widehat{\bar{\psi}} \star (i\gamma^\mu D_\mu - m)\widehat{\psi} - \frac{1}{4}\int d^4x\, \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}\,,$$

is not a renormalizabile theory because of the fermionic loop contributions [46, 47, 48]. It would be interesting to see if additional terms present in the NC $SO(2,3)_\star$ gravity induced Electrodynamics (3.45) can improve this behaviour.

The NC $SO(2,3)_\star$ gravity model also enables one to introduce coupling with scalar and non-Abelian gauge fields. In this way, it is possible to progress towards generalizing Standard Model to a NC spacetime using the setup we described in this article.

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