T-duality and non-geometry*

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Abstract

The role of double space is essential in new interpretation of T-duality and consequently in an attempt to construct M-theory. The case of open string is missing in such approach because until now there has been no appropriate formulation of open string T-duality. We will consider here reconsideration of T-duality of the open string. This will allow us to introduce some geometric features in non-geometric theories.

1. Introduction

We will show that "restricted general coordinate transformations", which includes transformations of background fields but not include transformations of the coordinates, is the symmetry T-dual to the local gauge transformations [1]. This will enable us to introduced new term in the Lagrangian, with additional gauge field A_i^D (D denotes components with Dirichlet boundary conditions). It compensate non-fulfilment of the invariance under restricted general coordinate transformation on the end-points of open string [1], as well as standard gauge field A_a^N (N denotes components with Neumann boundary conditions) compensate non-fulfilment of the local gauge invariance on the end-points of open string. Using generalized procedure [2] we will perform T-duality of vector fields linear in coordinates. We show that gauge fields A_a^N and A_i^D are T-dual to ${}^\star A_D^a$ and ${}^\star A_N^i$ respectively.

We will introduce the field strength of T-dual non-geometric theories as derivative of T-dual gauge fields along both T-dual variable y_μ and its double $\tilde{y}_\mu$. Therefore, we introduce some new features of non-geometric theories, where field strength has both antisymmetric and symmetric parts [1].

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2. Closed string T-duality

Let us start from the closed string action [3, 4]

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}[x] + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}[x]\Big]\partial_\alpha x^\mu \partial_\beta x^\nu \,, \qquad (1)$$

where $G_{\mu\nu}$ is a space-time metric and $B_{\mu\nu}$ is Kalb-Ramond field.

Action principle $\delta S = 0$, beside equations of motion produces boundary conditions

$$\gamma^{(0)}_\mu(x)\delta x^\mu /_{\sigma=\pi} \quad - \quad \gamma^{(0)}_\mu(x)\delta x^\mu /_{\sigma=0} = 0 \,,$$

where

$$\gamma^{(0)}_\mu(x) \equiv \frac{\delta S}{\delta x'^\mu} = \kappa\Big(2B_{\mu\nu}\dot{x}^\nu - G_{\mu\nu}x'^\nu\Big) \,. \qquad (2)$$

2.1. Buscher T-duality procedure

Applying standard Buscher T-duality procedure [5] we can obtain T-dual action

$${}^\star S[y] = \kappa \int d^2\xi \; \partial_+ y_\mu \, {}^\star\Pi_+^{\mu\nu} \, \partial_- y_\nu = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \theta_-^{\mu\nu} \partial_- y_\nu \,, \qquad (3)$$

where T-dual background fields

$${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu} \,, \qquad (4)$$

are defined in terms of effective metric $G^E_{\mu\nu}$ and non-commutative parameter $\theta^{\mu\nu}$

$$G^E_{\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \qquad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu} \,. \qquad (5)$$

T-duality transformation of variables are

$$\partial_\pm x^\mu \cong -\kappa\theta_\pm^{\mu\nu}\partial_\pm y_\nu \,, \qquad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu \,, \qquad (6)$$

which in canonical form can be written as

$$\kappa\, x'^\mu \cong {}^\star\pi^\mu \,, \quad \pi_\mu \cong \kappa y'_\mu \,, \quad -\kappa\, \dot{x}^\mu \cong {}^\star\gamma^\mu_{(0)}(y) \,, \quad \gamma^{(0)}_\mu(x) \cong -\kappa\, \dot{y}_\mu \,. \qquad (7)$$

3. T-duality of local gauge symmetries

We are going to consider open string T-duality. We will insist that each term should have corresponding T-dual one. For example

$$\begin{array}{ccccc} S(x) & G_{\mu\nu} & B_{\mu\nu} & A_a^N & A_i^D \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ {}^\star S(y) & {}^\star G^{\mu\nu} & {}^\star B^{\mu\nu} & {}^\star A_D^a & {}^\star A_N^i\,. \end{array} \tag{8}$$

It is well known from the literature that coupling with Neumann fields has a form

$$S_{A^N} = 2\kappa \int d\tau (A_a^N \dot{x}^a /_{\sigma=\pi} - A_a^N \dot{x}^a /_{\sigma=0})\,. \tag{9}$$

But, coupling with Dirichlet fields is not known

$$S_{A^D} = 2\kappa \int d\tau (A_i^D (?)^i /_{\sigma=\pi} - A_i^D (?)^i /_{\sigma=0})\,. \tag{10}$$

To find it we will use analogy with Neumann case.

3.1. Zwiebach approach for coupling with Neumann fields

Action of closed string theory is invariant under local gauge transformations

$$\delta_\Lambda G_{\mu\nu} = 0\,, \qquad \delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu\,. \tag{11}$$

Due to the boundary term the open string theory is not invariant [6]

$$\delta_\Lambda S[x] = 2\kappa \int d\tau (\Lambda_a \dot{x}^a /_{\sigma=\pi} - \Lambda_a \dot{x}^a /_{\sigma=0})\,. \tag{12}$$

To obtain gauge invariant action we should add the term

$$S_{A^N}[x] = 2\kappa \int d\tau (A_a^N \dot{x}^a /_{\sigma=\pi} - A_a^N \dot{x}^a /_{\sigma=0})\,, \tag{13}$$

where newly introduced vector field A_a^N transforms with the same gauge parameter Λ_a

$$\delta_\Lambda A_a^N = -\Lambda_a\,. \tag{14}$$

3.2. What is T-dual to local gauge transformations?

In order to continue we will use statement from Refs.[7, 8, 9]. If variation of energy-momentum tensor $T_\pm$ can be written as

$$\delta\, T_\pm = \{\Gamma, T_\pm\}\,, \tag{15}$$

then corresponding transformation of background fields is target-space symmetry of the theory.

For $\Gamma \to \Gamma_\Lambda = 2\int d\sigma \Lambda_\mu \kappa x'^\mu$ we can obtain just local gauge transformations. T-dual to generator $\kappa x'^\mu$ is π_μ, so that T-dual to Γ_Λ is

$$\Gamma_\xi = 2\int d\sigma\, \xi^\mu \pi_\mu\,. \tag{16}$$

The corresponding transformations are

$$\delta_\xi G_{\mu\nu} = -2\left(D_\mu \xi_\nu + D_\nu \xi_\mu\right),$$

$$\delta_\xi B_{\mu\nu} = -2\,\xi^\rho B_{\rho\mu\nu} + 2\partial_\mu(B_{\nu\rho}\xi^\rho) - 2\partial_\nu(B_{\mu\rho}\xi^\rho)\,. \tag{17}$$

These transformations exactly have the form of general coordinate transformations (GCT), the symmetry transformations of the space-time action. Are these transformations symmetries of the σ-model action?

World-sheet action is scalar under GCT, so both closed and open string actions are invariant under GCT. To understand what is T-dual to local gauge transformations it is useful to make transformations of the background fields, metric tensor $G_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$, with parameter ξ_μ and transformations of the string coordinates x^μ with different parameter $\bar{\xi}^\mu$, $\delta x^\mu = \bar{\xi}^\mu$. Using the equation of motion we obtain

$$\delta_\xi S[x] = -2\int_{\partial\Sigma} d\tau (\xi_\mu - \bar{\xi}_\mu)\, G^{-1\mu\nu}\gamma_\nu^{(0)}(x)\,. \tag{18}$$

If we introduce residual general coordinate transformations (RGCT), which include the transformations of background fields but not include the transformations of the string coordinates x^μ, $\bar{\xi}_\mu/_{\sigma=\pi} = \bar{\xi}_\mu/_{\sigma=0} = 0$, we obtain

$$\delta_\xi S[x] = -2\int_{\partial\Sigma} d\tau \xi_\mu\, G^{-1\mu\nu}\gamma_\nu^{(0)}(x)\,. \tag{19}$$

Note that according to (7) $-\kappa\dot{x}^\mu$ and $\gamma_\mu^{(0)}(x)$ are expressions T-dual to each other. So, local gauge transformations and RGCT are connected by T-duality.

4. Open string T-duality

Gauge invariant action for open string is [1]

$$S_{open}[x] = \kappa \textstyle\int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu$$
$$+ \quad 2\kappa \textstyle\int_{\partial\Sigma} d\tau \Big[A_a^N[x]\dot{x}^a - \frac{1}{\kappa}A_i^D[x]G^{-1ij}\gamma_j^{(0)}(x)\Big]\,, \tag{20}$$

where in literature $A_a^N[x]$ is known as massless vector field on Dp-brane and $A_i^D[x]$ as massless scalar oscillations orthogonal to the Dp-brane.

Note that gauge invariant and physical variables are

$$\begin{aligned} \mathcal{B}_{ab} &= B_{ab} + F_{ab}^{(a)}\,, \qquad \mathcal{G}_{ab} = G_{ab}\,, \\ \mathcal{B}_{ij} &= B_{ij} - 2A_D^k B_{kij} - F_{ij}^{(s)}(\hat{A}^D)\,, \\ \mathcal{G}_{ij} &= G_{ij} + F_{ij}^{(s)}(A^D)\,, \end{aligned} \tag{21}$$

where we introduced field strengths

$$F_{ab}^{(a)} = \partial_a A_b^N - \partial_b A_a^N\,, \qquad F_{ij}^{(s)}(A^D) = -2(\partial_i A_j^D + \partial_j A_i^D)\,, \tag{22}$$

and define

$$\hat{A}_i = B_{ij} G^{-1jk} A_k^D\,. \tag{23}$$

4.1. T-dual background fields of the open string

We will choose background vector fields linear in coordinates [1]

$$B_{\mu\nu} = const\,, \qquad G_{\mu\nu} = const\,, \tag{24}$$

$$A_a^N(x) = A_a^0 - \frac{1}{2}F_{ab}^{(a)}x^b\,, \qquad A_i^D(x) = A_i^0 - \frac{1}{4}F_{ij}^{(s)}x^j\,, \tag{25}$$

so that corresponding field strengths are constant. These forms of background fields satisfies space-time equations of motion for open string [10].

It is important to know that action depends on the coordinate x^μ itself and not only on its derivatives with respect to τ and σ. So, part with $A_i^D(x)$ does not have global shift symmetry, because the expression $\gamma_i^{(0)}$ contain x'^j which is not total derivative with respect to integration variable τ. So, we should apply T-dualization procedure [2] which work in absence of global symmetry.

T-dual background fields in terms of initial ones are

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\,, \tag{26}$$

$$^\star A_D^a(V) = G_E^{-1ab} A_b^N(V)\,, \quad ^\star A_N^i(V) = G^{-1ij} A_j^D(V)\,, \tag{27}$$

where

$$V^\mu = -\kappa\,\theta^{\mu\nu} y_\nu + G_E^{-1\mu\nu}\,\tilde{y}_\nu\,, \tag{28}$$

and

$$\tilde{y}_\mu \equiv -\varepsilon_\alpha{}^\beta \int d\xi^\alpha \partial_\beta y_\mu = \int (d\tau y'_\mu + d\sigma \dot{y}_\mu)\,. \tag{29}$$

Note that T-duality interchange Neumann with Dirichlet gauge fields.

4.2. Relation with standard approach

Up to gauge transformation we have

$$ {}^\star A_D^a = G_E^{-1ab}\left(A_b^N + \frac{1}{2}y_a\right) , \quad {}^\star A_N^i = G^{-1ij}A_j^D . \tag{30}$$

In standard approach one can not recognize Dirichlet vector fields. So if we put $A_i^D = 0$ and ${}^\star A_D^a = 0$ we obtain

$$ {}^\star A_N^i = 0 , \qquad y_a = -2A_b^N . \tag{31}$$

These are consistency conditions of standard approach.

5. The field strength for non-geometric theories

The particular form of V^μ from Eq.(28) implies features of non-geometric theories, see for example [11]. It produces non-commutativity and non-associativity of closed string coordinates.

In geometric theories the field strength for Abelian vector field is simple $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Because in non-geometric theories the vector field depends on V^μ, we expect that T-dual field strength will contain derivatives with respect to both variables y_μ and $\tilde{y}_\mu$. How to define the field strength for non-geometric theories?

For Neumann vector fields in initial theory we have

$$ S_A^N[x] = 2\kappa\int_{\partial\Sigma} d\tau A_a^N(x)\dot{x}^a = \kappa\int_\Sigma d^2\xi\partial_+ x^a \mathcal{F}_{ab}\,\partial_- x^b , \tag{32}$$

where only antisymmetric part contributes

$$ \mathcal{F}_{ab} = F_{ab}^{(a)} = \partial_a A_b^N(x) - \partial_b A_a^N(x) . \tag{33}$$

We are going to generalize such relation to non-geometric theories. For Dirichlet vector fields in initial theory we find

$$ \begin{aligned} S_A^D[x] &= 2\kappa\int_{\partial\Sigma} d\tau\Big(-\frac{1}{\kappa}A_i^D(x)G^{-1ij}\gamma_j^{(0)}(x)\Big) \\ &= 2\kappa\int_{\partial\Sigma} d\tau\Big(\mathcal{A}_{0i}[x]\dot{x}^i - \mathcal{A}_{1i}[x]x'^i\Big) = \kappa\int_\Sigma d^2\xi\,\partial_+ x^i\,\mathcal{F}_{ij}\,\partial_- x^j . \end{aligned} \tag{34}$$

Now, both antisymmetric and symmetric parts contribute

$$ \mathcal{F}_{ij} = \mathcal{F}_{ij}^{(a)} + \frac{1}{2}\mathcal{F}_{ij}^{(s)} , \tag{35}$$

where

$$ \mathcal{F}_{ij}^{(a)} = \Big[\partial_i\Big(2B_{jk}G^{-1kq}A_q^D\Big) - \partial_j\Big(2B_{ik}G^{-1kq}A_q^D\Big)\Big] , = \partial_i\mathcal{A}_{0j}(x) - \partial_j\mathcal{A}_{0i}(x) , \tag{36}$$

and

$$\mathcal{F}^{(s)}_{ij} = -2(\partial_i A^D_j + \partial_j A^D_i) = 2\Big(\partial_i \mathcal{A}_{1j}(x) + \partial_j \mathcal{A}_{1i}(x)\Big) \,. \tag{37}$$

For Dirichlet vector fields in T-dual theory we have

$$\begin{aligned} {}^\star S^D_A[y] &= 2\kappa \int_{\partial\Sigma} d\tau \Big(-\frac{1}{\kappa} {}^\star A^a_D(V)\, {}^\star G^{-1}_{ab}\, {}^\star \gamma^b_{(0)}(y) \Big) \\ &= \kappa \int_\Sigma d^2\xi \partial_+ y_a\, {}^\star \mathcal{F}^{ab}\, \partial_- y_b \,, \end{aligned} \tag{38}$$

with

$${}^\star \mathcal{F}^{ab} = {}^\star \mathcal{F}^{ab}_{(a)} + \frac{1}{2}\, {}^\star \mathcal{F}^{ab}_{(s)} \,. \tag{39}$$

For Neumann vector fields in T-dual theory

$${}^\star S^N_A[y] = 2\kappa \int_{\partial\Sigma} d\tau \Big({}^\star A^i_N(V) \dot{y}_i \Big) = \kappa \int_\Sigma d^2\xi\, \partial_+ y_i\, {}^\star \mathcal{F}^{ij}\, \partial_- y_j \,, \tag{40}$$

where

$${}^\star \mathcal{F}^{ij} = {}^\star \mathcal{F}^{ij}_{(a)} + \frac{1}{2}\, {}^\star \mathcal{F}^{ij}_{(s)} \,. \tag{41}$$

In Dirichlet case for non-geometric theories we obtain antisymmetric

$${}^\star \mathcal{F}^{ab}_{(a)} = -\kappa^2 \theta^{ac} F^{(a)}_{cd}\, \theta^{db} - G^{-1ac}_E F^{(a)}_{cd}\, G^{-1db}_E \,, \tag{42}$$

and symmetric field strengths

$${}^\star \mathcal{F}^{ab}_{(s)} = -2\kappa \Big[G^{-1ac}_E F^{(a)}_{cd}\, \theta^{db} + \theta^{ac} F^{(a)}_{cd}\, G^{-1db}_E \Big] \,, \tag{43}$$

while in Neumann case we have

$${}^\star \mathcal{F}^{ij}_{(a)} = -\frac{\kappa}{4} \left(\theta^{ik} F^{(s)}_{kq} G^{-1qj} + G^{-1ik} F^{(s)}_{kq} \theta^{qj} \right) \,, \tag{44}$$

and

$${}^\star \mathcal{F}^{ij}_{(s)} = -\frac{1}{2} \left(G^{-1ik}_E F^{(s)}_{kq} G^{-1qj} + G^{-1ik} F^{(s)}_{kq} G^{-1qj}_E \right) \,. \tag{45}$$

Finally, we should write out expressions for T-dual field strengths ${}^\star\mathcal{F}^{\mu\nu}$ in terms of derivative of T-dual gauge fields ${}^\star\mathcal{A}^a_0(V)$ and ${}^\star\mathcal{A}^a_1(V)$ with respect to variables y_μ and $\tilde{y}_\mu$. If we define $y^\alpha_\mu = \{y^0_\mu = y_\mu,\, y^1_\mu = -\tilde{y}_\mu\}$ and $\partial^\mu_\alpha \equiv \frac{\partial}{\partial y^\alpha_\mu} = \{\frac{\partial}{\partial y_\mu}\,, -\frac{\partial}{\partial \tilde{y}_\mu}\}$ we find

$$\begin{aligned} {}^\star \mathcal{F}^{\mu\nu} &= {}^\star \mathcal{F}^{\mu\nu}_{(a)} + \tfrac{1}{2} {}^\star \mathcal{F}^{\mu\nu}_{(s)} \\ &= \eta^{\alpha\beta} \Big[\partial^\mu_\alpha\, {}^\star \mathcal{A}^\nu_\beta(V) - \partial^\nu_\alpha\, {}^\star \mathcal{A}^\mu_\beta(V) \Big] - \varepsilon^{\alpha\beta} \Big[\partial^\mu_\alpha\, {}^\star \mathcal{A}^\nu_\beta(V) + \partial^\nu_\alpha\, {}^\star \mathcal{A}^\mu_\beta(V) \Big] \,. \end{aligned} \tag{46}$$

We can check this expression in other way

$$
\begin{aligned}
{}^\star S_A[y] = {}^\star S_A^D[y] + {}^\star S_A^N[y] &= 2\kappa\eta^{\alpha\beta}\int_{\partial\Sigma} d\tau^\star\mathcal{A}^\mu_\alpha[V]\,\partial_\beta y_\mu \\
&= \kappa\int_\Sigma d^2\xi\partial_+ y_\mu{}^\star\mathcal{F}^{\mu\nu}\partial_- y_\nu\,. \qquad (47)
\end{aligned}
$$

6. Conclusions

The expression (46) we can consider as a general definition of the field strength for non-geometric theories. Beside antisymmetric part ${}^\star\mathcal{F}^{\mu\nu}_{(a)}$ it also contains the symmetric one ${}^\star\mathcal{F}^{\mu\nu}_{(s)}$. In definition of both parts, derivatives with respect to both T-dual coordinate y_μ and to its double $\tilde{y}_\mu$ contribute.

The unusual form of ${}^\star\mathcal{F}^{\mu\nu}$ is a consequence of two facts:

1. the T-dual vector field ${}^\star A^a_D(V)$ are not multiplied by $\dot{y}_a$ but with T-dual σ-momentum ${}^\star G^{-1\star}_{ab}\gamma^b_{(0)}$.

2. the T-dual vector fields depend on V^μ which is function on both y_μ and $\tilde{y}_\mu$.

References

[1] B. Sazdović, *From geometry to non-geometry via T-duality*, ArXiv 1606.01938.

[2] Lj. Davidović and B. Sazdović, *JHEP* **11** (2015) 119.

[3] J. Polchinski, *String theory*, Cambridge University Press, 1998.

[4] K. Becker, M. Becker and J. Schwarz, *String Theory and M-Theory: A Modern Introduction*, Cambridge University Press, 2007.

[5] T. Buscher, *Phys. Lett.* **B 194** (1987) 51; **201** (1988) 466.

[6] B. Zwiebach, *A First Course in String Theory*, Cambridge University Press, 2004.

[7] M. Evans and B. A Ovrut, *Phys.Rev.* **D 39** (1989) 3016.

[8] M. Evans and B. A Ovrut, *Phys.Rev.* **D 41** (1990) 3149.

[9] Lj. Davidović and B. Sazdović, in preparation.

[10] R. G. Leigh, *Mod. Phys. Lett.* **A 4** (1989) 2767.

[11] Lj. Davidović, B. Nikolić and B. Sazdović, *EPJ* **C 75** (2015) 576.

[12] Lj. Davidović and B. Sazdović, *EPJ* **C 74** (2014) 2683.

[13] B. Sazdović, *JHEP* **08** (2015) 055.