Holography for Quantum Quenches
(Thermalization after holographic bilocal quench)

Irina Aref’eva

joint works with M. A. Khramtsov and M. D. Tikhanovskaya,
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Steklov Mathematical Institute, Russian Academy of Sciences

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Plan

• Introduction
• Thermalization after holographic bilocal quench
• Thermalization after global quench in thermal media
• Conclusion [comparison of local/global quenches]


**Introduction. Thermalization in a quantum system**

**Thermalization** in a quantum system is a major theoretical challenge. It is involved in many problems of physics (and not only) which involve initial states which are out of equilibrium:

- Early Universe
- Heavy ion collisions
- Dynamics of cold atomic gas
- etc.
Introduction. Thermalization/equilibration after quantum quench

A natural setup to study thermalization in closed quantum systems is quantum quench:

- Quantum system with a Hamiltonian $H$;
- At time $t = 0$, the Hamiltonian parameter is changed abruptly $H \rightarrow H'$ and $H'\ket{\psi'} = E\ket{\psi'}$
- for times $t > 0$ system evolves with $H$, and $\ket{\psi(t)} = e^{-iH_2t}\ket{\psi'}$.

Let $A$ be a subsystem, with density matrix $\rho_A(t) = \text{Tr}_{\bar{A}}\ket{\psi(t)}\bra{\psi(t)}$.

The system thermalizes, if for any subsystem $A$ it is true that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_A(t) dt = \rho_\beta = \frac{1}{Z}e^{-\beta H_2} \text{ for some } \beta
$$

How do we probe it?

- Entanglement entropy: $S(A) = -\text{Tr}_A \rho_A \log \rho_A$ - our primary tool
- More "fine-grained" observables: Renyi entropies, correlation functions of specific operators, . . . .
Quenches in CFT

CFT is a convenient arena to study quench dynamics (Cardy, Calabrese,’05)

- **Global quench** - excites every point of the circle uniformly in the initial state - **popular setup for studies of thermalization**:

- **Local quench**: excites a point of the circle in the initial state

- **Bilocal quench**: excites two antipodal points (AKT: 1706.07390)

The main goal of our work is to study the non-equilibrium dynamics of entanglement during thermalization after the bilocal quench in (1 + 1)d CFT on a cylinder using the holographic duality.
The bulk dual of the bilocal quench

- Local excitations in the boundary \( \text{CFT}_2 \) produce massless particles in the bulk ⇒
  The holographic dual is the \( \text{AdS}_3 \) spacetime with two colliding massless point particles.
- We are interested in thermalization ⇒ we study the case when the colliding particles **produce a black hole**.
The AdS$_3$ spacetime

- a hyperboloid in the 4D flat space: $x_0^2 + x_3^2 - x_1^2 - x_2^2 = 1$
  Parametrize the hyperboloid by global coordinates:

  $x_3 = \cosh \chi \cos t, \quad x_0 = \cosh \chi \sin t, \quad x_1 = \sinh \chi \cos \phi, \quad x_2 = \sinh \chi \sin \phi$

  where $\chi \in \mathbb{R}_+, \ t \in [-\pi, \pi], \ \phi \in [0, 2\pi]$. The metric is

  $$ds^2 = -\cosh^2 \chi\, dt^2 + d\chi^2 + \sinh^2 \chi\, d\phi^2$$

- $SL(2, \mathbb{R})$ group manifold:

  $$X = \begin{pmatrix} x_3 + x_2 & x_0 + x_1 \\ x_1 - x_0 & x_3 - x_2 \end{pmatrix}, \quad \det X = x_0^2 + x_3^2 - x_1^2 - x_2^2 = 1$$

- solution of vacuum 3D Einstein equations with negative cosmological constant

  Identification isometries: $X \rightarrow X^* = u^{-1} X u; \ u - holonomy\ of\ the\ defect$
Massless particle in AdS$_3$

The holonomy is: \( u_{\text{massless}} = 1 + p^\mu \gamma_\mu \), \hspace{1cm} (Matschull, gr-qc/9809087)

where

\[
\begin{align*}
1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]
BTZ black hole in global AdS$_3$

Figure: The maximally extended BTZ black hole in global coordinates.

Holonomy: $u_{\text{BTZ}} = \begin{pmatrix} -\cosh \mu & \sinh \mu \\ \sinh \mu & -\cosh \mu \end{pmatrix}$; where $\mu = \pi R$ - mass of BH
Collision of particles in the center of mass frame

A. 

B. 

C. 

D.
Collision of massless particles in the BTZ rest frame

\[ ds^2 = -\cosh^2 \chi \, dt^2 + d\chi^2 + \sinh^2 \chi \, d\phi^2 \]

Identified surfaces:

\[ W_\pm : \tan \chi (\mathcal{E} \cos \phi \pm \sin \phi) = -\mathcal{E} \sin \tau, \quad \mathcal{E} = \coth \frac{R\pi}{2} \]

\[ V_\pm : \quad \tanh \chi \sin \phi = \mp \sin t \tanh \pi R \]

Figure: Collision of particles in the BTZ rest frame, Matschull
Black hole creation in BTZ coordinates

To do the holographic computation, we need to map this quotient of global AdS$_3$ to an asymptotically AdS$_3$ spacetime with cylindrical boundary. For this, we make transition to BTZ-Schwarzshild coordinates:

\begin{align*}
x_0 &= \cosh \chi \sin \tau = -\frac{r}{R} \cosh R \varphi \\
x_2 &= \sinh \chi \sin \phi = \frac{r}{R} \sinh R \varphi \\
x_3 &= \cosh \chi \cos \tau = \sqrt{\frac{r^2}{R^2} - 1} \sinh R \ t \\
x_1 &= \sinh \chi \cos \phi = \sqrt{\frac{r^2}{R^2} - 1} \cosh R \ t
\end{align*}

The metric has the form \( ds^2 = -(r^2 - R^2) \ dt^2 + \frac{dr^2}{r^2 - R^2} + r^2 \ d\varphi^2 \), where \( r > R, \varphi \in \mathbb{R}, \ t \in [0, +\infty) \).

What happens to the identification?
Black hole creation in BTZ coordinates

Figure: Cartoon of black hole creation in BTZ coordinates (A. Jevicki, J. Thaler, hep-th/0203172). A: particles start from the boundary at \( t = 0 \). B: Particles move through the bulk towards each other. C: Particles asymptotically approach the horizon.

\[
\begin{align*}
\text{ds}^2 &= -(r^2 - R^2) \, dt^2 + \frac{dr^2}{r^2 - R^2} + r^2 \, d\varphi^2 , \\
\text{Identified surfaces:} \\
\mathcal{W}_\pm : \quad \tanh \chi \frac{\sin \phi}{\sin \tau} &= - \tanh R \varphi ; \\
V_- \sim V_+ &\iff \varphi \sim \varphi + 2\pi .
\end{align*}
\]
Two coordinate systems

Figure: A. 3D picture of identification surfaces in global AdS$_3$. B. Creation of the black hole by colliding particles in BTZ coordinates. Red surfaces are the faces of identification $W_{\pm}$. 
Holographic entanglement entropy

The entanglement entropy of a subsystem on a spatial subregion $A$ in the boundary theory is calculated according to the formula

$$S(A) = \frac{A}{4G}; \quad (1)$$

where $A$ is the minimal area of a co-dimension 2 extremal surface in the bulk which has the same boundary as $A$ and is homologous to $A$.

**AdS$_3$/CFT$_2$ case.** In $d = 2$ $A = [a, b]$ is a segment of the circle. To calculate HEE, one has to find the minimal geodesic between equal-time points $a$ and $b$ on the boundary and calculate its length:

$$S(a, b) = \frac{L_{\text{min}}(a, b)}{4G}; \quad \text{where} \quad G = \frac{3}{2c}. \quad (2)$$
Denote
\[ \Delta t = t_b - t_a ; \quad t_0 = \frac{1}{2}(t_b + t_a) ; \quad \Delta \varphi = \varphi_b - \varphi_a ; \quad \varphi_0 = \frac{1}{2}(\varphi_b + \varphi_a) ; \]

- **Direct geodesics** do not go through identification surfaces \( W_\pm \).
- **Crossing geodesics** go through the identification surfaces \( W_\pm \).
Two patterns of entanglement:

(i) HEE is constant for orange segments with $\varphi_a, \varphi_b \in [-\pi,0)$, or $\varphi_a, \varphi_b \in [0,\pi)$

(ii) HEE grows with time up to thermal value for blue segments with $\varphi_a \in [-\pi,0)$, and $\varphi_b \in [0,\pi)$:

Denote $\Delta \varphi = \varphi_b - \varphi_a$, $\varphi_0 = \frac{1}{2}(\varphi_a + \varphi_b)$. 
Holographic entanglement entropy: results

- Static thermal equilibrium regime: direct geodesic dominates.

\[ S_{\text{eq}}(a, b) = \frac{c}{3} \log \left( \frac{2}{\epsilon} \sinh \left( R \frac{\Delta \varphi}{2} \right) \right); \]

This is thermal HEE.

- Dynamic non-equilibrium regime: crossing geodesic dominates.

\[ S_{\text{non-eq}}(a, b | t) = \frac{c}{6} \log \left[ \frac{2}{\epsilon} \left( -1 + \mathcal{E}^2 + (1 + \mathcal{E}^2) \cosh R \Delta \varphi + \mathcal{E}^2 \cosh 2R \varphi_0 + \mathcal{E}^2 \cosh 2R t - 2 \mathcal{E} \sinh R \Delta \varphi + 4 \mathcal{E} \cosh R t \cosh R \varphi_0 \left( \sinh R \frac{\Delta \varphi}{2} - \mathcal{E} \cosh R \frac{\Delta \varphi}{2} \right) \right] \].

For \( \varphi_0 = 0 \): symmetry between \( t \) and \( \frac{\Delta \varphi}{2} \)
Evolution of entanglement entropy

**Figure**: Here red curve is the function $\Delta S(t) = S_{\text{non-eq}}(t) - S_{\text{eq}}$, green dashed curve is the quadratic approximation, and blue dashed line is the linear asymptotic. **Main feature** - sharp transition to saturation.
Evolution of entanglement entropy [”global heating up”]

\[ \Delta S(t) = S_{\text{non-eq}}(t) - S_{\text{eq}}, \]

blue: \( z_H = \infty, z_h = 1 \);
green \( z_H = 2.5, z_h = 1 \);
red \( z_H = 1.3, z_h = 1 \). Top curves correspond to \( \ell = 7 \) and bottom ones to \( \ell = 3.5 \).

Main feature - smooth transition to saturation.

Time scales of entanglement equilibration

- ”Pre-local equilibration time” $t = t_1 = \frac{\Delta \varphi}{2R}$
  (H.Liu, S.Suh,1305.7244; I.A., D.Ageev, 1704.07747, in the global quench context )
- Crossing the apparent horizon: $t = t_2$
- Thermalization time $t = t^{(a,b)}_* -$ HEE saturates

$$\cosh R \ t^{(a,b)}_* = \cosh R \varphi_0 \left( \cosh R \frac{\Delta \varphi}{2} - \frac{1}{\xi} \sinh R \frac{\Delta \varphi}{2} \right)$$

$$+ \sqrt{\cosh^2 R \varphi_0 \left( \cosh R \frac{\Delta \varphi}{2} - \frac{1}{\xi} \sinh R \frac{\Delta \varphi}{2} \right)^2 - \cosh^2 R \varphi_0 - \sinh^2 R \frac{\Delta \varphi}{2} + \frac{1}{\xi} \sinh R \Delta \varphi}$$

Unlike the global quench case, we have **sharp transition** to saturation
Evolution of entanglement entropy

1. **Early-time quadratic growth:** $t < t_1$

   \[ S_{\text{non-eq}}(a, b|t) = S_{\text{non-eq}}(a, b|0) + f(\varphi_a, \varphi_b)\ t^2 + O(t^4); \]

2. **Intermediate regime:** $t_1 \leq t < t_2$. Interpolation between quadratic and linear growth

3. **Linear growth:** $t_2 \leq t < t^*$

   \[ \Delta S(t) = S_{\text{non-eq}} - S_{\text{eq}} = \frac{c}{3} R \ t + \frac{c}{3} \log \left( \frac{\coth \frac{\pi R}{2}}{8 \sinh^2 R \frac{\Delta \varphi}{2}} \right) + O(e^{-Rt}). \]

4. **Saturation:** $t \geq t^*$. HEE is at thermal value.
Universality of entanglement growth

Figure: A: The shell in the BH background; B: $\Delta(t, \ell) S$; C: The holographic entanglement entropy $S(t, \ell)$ as function on $t$ for given $\ell$; D: $S(t, \ell)$ as function on $\ell$ for given $t$. 
Memory loss and entanglement tsunami [bilocal quench]

Figure: A: Entanglement spreading in case of symmetric intervals. The horizontal plateau represents the equilibrium regime. B: Density plot of non-equilibrium HEE as a function of $\varphi = \frac{\Delta \varphi}{2}$ and $U = t - \varphi$, with $R = 5$.

Tsunami=wave character=$S(\ell, t) = S(\ell - t)$
Memory loss and entanglement tsunami [global heating up]

Figure: 3D plot for $\Delta S(t, \ell)$ for the global heating up.
Bound on size of segments which probe the interior:

\[
\frac{\Delta \varphi}{2} \geq \varphi_{\text{hor}} = \frac{1}{2R} \arcsinh \tanh \frac{\pi R}{2}
\]
Evolution of mutual information (bilocal quench)

\[ I(t) \]

A.  

\[ I(A, B) = S(A) + S(B) - S(A \cup B) \]
Evolution of mutual information (global quench)

\[ I(A, B) = S(A) + S(B) - S(A \cup B) \]

**Figure:** Different regimes of the MI evolution in the heating process of two disjoint intervals.
Evolution of MI for composite systems (global quench)

5 types of MI time dependence

IA, O.Inozemchev, I.Volovich
Conclusions

- Non-trivial non-equilibrium dynamics are shown by subsystems which contain one of the excitations.
- Explicit formula for non-equilibrium HEE.
- Many similarities with global quench ⇒ more evidence for universality of entanglement growth
- Significant difference from the global quench:
  - sharp transition
  - $\phi < - > t$ symmetry
- Linear growth, loss of memory about the initial state and black hole interior are intimately connected:
  Linear growth of entanglement is a diagnostic of the information loss in the bulk
- Mutual information:
  - many possibilities;
  - "Bell"-type ("breather") in the temporal behavior.
Open questions

- CFT computation of HEE and correlation functions
- Corrections to the holographic limit (memory/information restoration?)
- Higher-dimensional generalizations
- Generalization to the $n$-local case (collision of $n$ particles in the bulk)
Backup slides
Geodesic approximation for two-point boundary correlators

We are interested in the two-point correlation function of light* scalar operator $\langle \mathcal{O}_\Delta(a) \mathcal{O}_\Delta(b) \rangle$. Consider the propagator for bulk field $\Phi$ on asymptotically AdS space in the worldline representation:

$$G(a, b) = \int_{X(0)=a}^{X(1)=b} \mathcal{D}X(\lambda) \ e^{im \int_0^1 d\lambda \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}}, \quad m^2 = \Delta(\Delta - d)$$

Steepest descent expansion ($m \sim \Delta \gg 0$):

$$G(a, b) \sim \sum e^{-\Delta\mathcal{L}(a,b)}$$ (2)

Extrapolate the bulk field to the boundary: $\mathcal{O}_\Delta = \lim_{\epsilon \to 0} \epsilon^{-\Delta} \Phi \rightarrow$

Geodesic approximation for boundary correlators (Balasubramanian, Ross, 2000):

$$\langle \mathcal{O}_\Delta(a) \mathcal{O}_\Delta(b) \rangle = \sum e^{-\Delta\mathcal{L}_{\text{ren}}(a,b)}$$
Geodesic image method

In our case there are multiple geodesics connecting two boundary points, so we have to sum over them

\[
\langle \Ocal_\Delta(a)\Ocal_\Delta(b) \rangle = \sum \text{e}^{-\Delta \mathcal{L}_{\text{ren}}(a,b)} = \sum Z_n(a, b^{*n}) \Delta \text{e}^{-\Delta \mathcal{L}_{\text{ren}}(a, b^{*n})}
\]

Sum includes direct geodesic and geodesics which wind around defects. The latter ones can be accounted for using image geodesics from point \(a\) to images of \(b\) w.r.t. identification isometry: i.e. \(x_{b^*} = u^{-1}x_b u\).


- Consider image geodesics: \(\mathcal{L}(a, b^*), \mathcal{L}(a, b^{**}), \ldots, \mathcal{L}(a, b^{*n})\)
- Length of a winding geodesic equals to the length of an image geodesic
- The renormalization scheme takes into account invariance with respect to identification \(*\).
- Applicability of semiclassical expansion: \(\mu(\varepsilon) \gg \Delta \gg 0\)
- The prescription is continued to the Lorentzian signature using the reflection mapping according to recipe in 1604.08905
Applicability of the geodesic prescription

- The background metric must have a well-defined Euclidean analytic continuation
- In the Lorentzian signature the prescription is viable only for spacelike-separated points on the boundary
- $0 \ll \Delta \ll \mu$. (Steepest descent and no backreaction)
- In general, one has to sum over all geodesics between two given boundary points
- In the general case, there is a non-perturbative contribution to the full propagator. It vanishes in the case where AdS is an orbifold (I. Aref’eva, M. K., arXiv:1601.02008)
- The renormalization scheme must be tailored for every specific deformation of AdS spacetime