Superstring compactification and Special Frobenius manifold structure.

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2017

Introduction.

For finding the low-energy Lagrangian of String theory compactified on a CY manifold, one needs to know the Special Kähler geometry on the moduli space of complex structure on a CY threefold X.

A way to compute this was proposed in the work by Candelas, de la Ossa,Green and Parkes (CDGP).
Kähler potential of the metric on the moduli space is expressed bilinearly in terms of periods $\in H_3(X)$ of the CY 3-form Ω.

Periods ω_μ in a particular basis of cycles in $q_\mu \in H_3(X)$ have been found by Berglund, Candelas, de la Ossa, Font, Hubsch, Jancic and Quevedo (BCDFHJQ) for a large number of CY manifolds

$$\omega_\mu := \oint_{q_\mu \in H_3(X)} \Omega,$$

The main difficulty in the CDGP approach is finding then a symplectic basis of periods Π_μ.

We suggest an alternative way to the computation of Kähler geometry for the case CY manifold given by a hypersurface $W_0(x) = 0$ in a weighted projective space.

Actually it is not necessary to look for the symplectic basis of periods. The Kähler potential can be expressed in terms the periods $\omega^{\pm}_{\mu}(\phi)$ and their intersection matrix $C_{\mu\nu} = q_\mu \cap q_\nu$ as

$$e^{-K(\phi)} = \omega_\mu(\phi) C^{\mu\nu} \bar{\omega}_\nu(\phi).$$

To find the intersection matrix $C_{\mu\nu}$ we use the fact that the moduli space of CY manifold is a subspace in the Special Frobenius manifold (SFM) which arises on the deformations of the singularity defined by the Landau-Ginzburg superpotential $W_0(x)$.

This fact allows to express $C_{\mu\nu}$ in terms the holomorphic metric $\eta_{\mu\rho}$ on the Special Frobenius manifold.

Also using the relation with the Special Frobenius manifold we can define and find explicitely an additional basis of periods called σ_μ connected by some constant matrix T_μ^ν with the periods $\omega_\mu^\pm(\phi)$

$$\omega_\mu^\pm(\phi) = T_\mu^\nu \, \sigma_\nu^\pm(\phi).$$

The matrix T_μ^ν can be used for expressing $C_{\mu\nu}$ in terms of the FM metric $\eta_{\mu\rho}$.

Kähler potential is then given in terms of the periods σ_μ, the holomorphic FM metric $\eta_{\mu\rho}$ and the matrix T_μ^ν as follows:

$$e^{-K} = \sigma_\mu^+ \eta^{\mu\rho} M_\rho^\nu \overline{\sigma_\nu^-}, \qquad M = T^{-1}\bar{T}.$$

Below we prove this formula and show how it can be applied to the different Calabi-Yau manifolds.

Special geometry

Recall the basic facts about the special Kähler geometry and how it arises on the CY moduli space (BGH-S-CD).

Let moduli space $\mathcal{M}$ of complex structures of a given CY manifold is n-dimensional and $z^1 \cdots z^{n+1}$ are the special (projective) coordinates on it.
Then there exists a holomorphic homogeneous function $F(z)$ of degree 2 in z called a prepotential such that the Kähler potential $K(z)$ of the moduli space metric is given by

$$e^{-K(z)} = z^a \cdot \frac{\partial \bar{F}}{\partial \bar{z}^{\bar{a}}} - \bar{z}^{\bar{a}} \cdot \frac{\partial F}{\partial z^a}$$

This metric on the moduli space of complex structures is a metric that naturally arises from deWitt (Polyakov) metric on a space of metrics on CY manifolds.

Special geometry on moduli spaces

Let X is CY three-fold and y^μ $(\mu = 1, 2, 3)$ are complex coordinates on X.

The moduli space of X is the space of metric perturbations of X that preserve Ricci-flatness.

The metric on the complex structure CY moduli space obtained from natural metric for CY metric deformations of type $\delta_a g_{\mu\nu},\ \delta_{\bar{b}} g_{\bar{\mu}\bar{\nu}}$ preserving Ricci-flatness is

$$G_{a\bar{b}} = \int_X \mathrm{d}^6 y\, g^{1/2}\, g^{\mu\bar{\sigma}} g^{\nu\bar{\rho}} \delta_a g_{\mu\nu} \delta_{\bar{b}} g_{\bar{\sigma}\bar{\rho}}.$$

The deformations which leave the metric Ricci flat corresponds to elements in $H^{2,1}(X)$:

$$\delta_a g_{\bar{\alpha}\bar{\beta}} \to \chi_{a,\mu\nu\bar{\beta}} \sim \Omega_{\mu\nu\lambda} g^{\lambda\bar{\alpha}} \delta_a g_{\bar{\alpha}\bar{\beta}}$$

We can then rewrite the above metric as

$$G_{a\bar{b}} = \frac{\int_X \chi_a \wedge \bar{\chi}_{\bar{b}}}{\int_X \Omega \wedge \bar{\Omega}}.$$

$a, \bar{b}$ are indices of complex coordinates in the deformation space.

From Kodaira Lemma:

$$\partial_a \Omega = k_a \Omega + \chi_a,$$

it follows that this metric is a Kähler :

$$G_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \ln \int_X \Omega \wedge \bar{\Omega}$$

To obtain the bilinear formulae written above, define the basis of periods as integrals over Poincare dual symplectic basises $A^a, B_b \in H_3(X, \mathbb{Z})$:

$$A^a \cap B_b = \delta^a_b, \qquad A^a \cap A^b = 0, \quad B_a \cap B_b = 0.$$

With this, we define periods in the symplectic basis

$$z^a = \int_{A^a} \Omega, \; F_b = \int_{B_b} \Omega.$$

and obtain

$$e^{-K} = \int_X \Omega \wedge \bar{\Omega} = z^a \cdot \bar{F}_{\bar{a}} - \bar{z}^{\bar{a}} \cdot F_a.$$

From the same Lemma we obtain

$$\int_X \Omega \wedge \partial_a \Omega = F_a - z^b \partial_a F_b = 0.$$

It follows

$$F_a(z) = \frac{1}{2} \partial_a F(z),$$

and

$$e^{-K(z)} = z^a \cdot \frac{\partial \bar{F}}{\partial \bar{z}^{\bar{a}}} - \bar{z}^{\bar{a}} \cdot \frac{\partial F}{\partial z^a}$$

where $F(z) = 1/2 z^b F_b(z)$.
That is the Kähler potential is expressed through the prepotential F. So the geometry of the moduli space of CY threefold is special.
Using the notation $(\Pi_\mu) = \left(\frac{\partial F}{\partial z^a}, z^a\right)$ for the vector of periods, we have

$$e^{-K(z)} = \Pi_\mu \Sigma^{\mu\nu} \bar{\Pi}_\nu,$$

where symplectic unit Σ is the inverse intersection matrix for cycles of the symplectic basis .
We can rewrite this expression in an arbitrary basis of cycles as

$$e^{-K(\phi)} = \omega_\mu(\phi) C^{\mu\nu} \bar{\omega}_\nu(\phi).$$

CY as a hypersurface in a weighted projective space

We will concentrate on the case when Calabi-Yau manifold is defined as a hypersurface in a weighted projective space.
Let $x_1, \ldots, x_5$ be homogeneous coordinates in a weighted projective space $\mathbb{P}^4_{(k_1,\ldots,k_5)}$ and

$$X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} | \ W_0(x) = 0\}.$$

$W_0(x)$ is some quasi-homogeneous polynomial, that defines an isolated singularity in the origin

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x_i)$$

and

$$\deg W_0(x) = d = \sum_{i=1}^{5} k_i.$$

The last relation ensures that X is a Calabi-Yau manifold.

The moduli space of complex structures on Calabi-Yau threefold X is then given by homogeneous polynomial deformations of this singularity modulo coordinate transformations:

$$W(x,\phi) = W_0(x) + \phi_0 \prod x_i + \sum_{s=0}^{\mu} \phi^s e_s(x),$$

$e_s(x)$ are polynomials of x with the same weight as $W_0(x)$.
The holomorphic 3-form Ω is given as a residue of a 5-form in the underlying affine space $\mathbb{C}^5$:

$$\Omega = \frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial W(x)/\partial x_4} = \frac{1}{2\pi i} \oint_{|x_5|=\delta} \mathrm{Res}_{W(x)=0} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_5}{W(x)}.$$

A special basis of periods

Having the explicit expression for Ω, we can define a special basis of periods $\omega_\mu(\phi)$ as follows (CDGP).
We choose the so-called cycle q_1, which is a torus in the large complex structure limit $\phi_0 >> 1$ (for simplicity other $\phi^s = 0$):

$$W(x,\phi) = W_0(x) + \phi_0 \prod x_i \ .$$

In this limit, we can define an 5−dimensional torus $Q_1 = |x_i| = \delta_i$ surrounding the hypersurface $W(x) = 0$ in $\mathbb{C}^5$. It corresponds to an 3-dimensional torus $q_1 \subset X$. Then the fundamental period is

$$\omega_1(\phi) := \int_{q_1} \Omega = \int_{Q_1} \frac{\mathrm{d}x^1 \cdots \mathrm{d}x^5}{W(x,\phi)}$$

and is given by a residue in its large ϕ_0 expansion.
More periods ω_μ may be obtained as analytic continuations of ω_1 in ϕ. This can be done by continuing $\omega_1(\phi)$ for a small ϕ_0 using Barnes' trick and using the symmetry of $W_0(x)$ afterwards.

Namely, there is a group of *phase* symmetries Π_X acting diagonally on x_i and preserving $W_0(x)$.
When $W_0(x)$ is deformed, the group actions can be extended on a parameter space with an action $\mathcal{A}$ such that

$$W(g \cdot x,\ \mathcal{A}(g) \cdot \phi_0) = W(x, \phi).$$

The moduli space is then at most a factor of the parameter space $\{\phi^s\}/\mathcal{A}$.
This allows defining a set of other periods as,

$$\omega_{\mu_g}(\phi) = \omega_1(\mathcal{A}(g) \cdot \phi_0), \ \ g \in G_X$$

In many cases this construction gives the whole basis of periods for the manifold X.

Periods as oscillatory integrals

The next important step is to transform the integrals for the periods $\int_{q_\mu} \Omega$ to the oscillatory form. Starting from

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega = \int_{Q_\mu} \frac{\mathrm{d}^5 x}{W(x)},$$

where $q_\mu \in H_3(X)$ and $Q_\mu \in H_5(\mathbb{C}^5 \backslash W(x) = 0)$,
we can present them in the form

$$\int_{Q_\mu} \frac{\mathrm{d}^5 x}{W(x)} = \int_{Q_\mu^\pm} e^{\mp W(x)} \mathrm{d}^5 x$$

where $Q_\mu^\pm \in H_5(\mathbb{C}^5, \ \mathrm{Re} W_0(x) = \pm\infty)$.
The map $Q_\mu \to Q_\mu^\pm$ is possible is due to the isomorphism

$$H_3(X) \to H_5(\mathbb{C}^5 \backslash W(x) = 0) = H_5(\mathbb{C}^5, \ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d \cdot \mathbb{Z}}$$

So, more precisely $Q_\mu^\pm \in H_5(\mathbb{C}^5, \ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d \cdot \mathbb{Z}}$ which is a subgroup of $H_5(\mathbb{C}^5, \ \mathrm{Re} W_0(x) = \pm\infty)$ defined below.

Special Frobenius manifold.

Let the polynomial $W_0(x)$ in $\mathbb{C}^5$, which defines CY hypersurface in 4-dimensional weighted projective space, is a quasi-homogeneous polynomial:

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x)$$

For Quintic threefold $W_0(x) = 1/5(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5)$.
$W_0(x)$ has an isolated singularity in the origin $\mathbb{C}^5$.
The connection with the singularity theory is important for computing the Special geometry of the CY moduli space.
Consider the Milnor ring of this singularity

$$R_0 = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\partial_1 W_0(x) \cdot \ldots \cdot \partial_5 W_0(x)}.$$

Actually, we need to consider not whole Milnor ring but its $Q = Z_d$-invariant Special subring R_0^Q.
This subring is generated by marginal deformations of the singularity, which have the same weight as $W_0(x)$ and correspond to moduli of CY. That is weights of elements of R_0^Q are integer multiples of d.

We will denote as $e_s(x)$ (with latin indexes s) elements that correspond to the complex structure deformations of M_c and as $e_\mu(x)$ (with greek indexes μ) elements of the basis of $R(W_0)$. Dimension of the subring R_0^Q is equal to the dimension of $H_3(X)$, where X is Calabi-Yau defined by W_0.
In Quintic case the dimension of Milnor ring$= 1024$, $dim R_0^Q = dim H_3(X) = 204$ and $dim M_c = 101$.
There exist a natural multiplication with structure constants $C_{\mu\nu}^\sigma$ in R_0^Q and a pairing $\eta_{\mu\nu}$, turning R_0^Q into Frobenius algebra.

$$\eta_{\mu\nu} = \mathrm{Res} \frac{e_\mu \cdot e_\nu}{\partial_1 W_0(x) \cdots \partial_5 W_0(x)},$$
$$C_{\mu\nu\lambda} = C_{\mu\nu}^\sigma \eta_{\sigma\lambda} = \mathrm{Res} \frac{e_\mu \cdot e_\nu \cdot e_\lambda}{\partial_1 W_0(x) \cdots \partial_5 W_0(x)}.$$

Consider the space of deformations of this singularity

$$W(x) = W_0(x) + \sum t^\mu e_\mu(x).$$

where $e_\mu(x) \in R_0^Q$.

On the space with parameters t^{μ} arises the structure of Special Frobenius manifold $\mathcal{M}_F$ with multiplication structure constants $C^{\rho}_{\mu\nu}(t)$ for the ring R^Q defined by the deformed singularity $W(x)$

$$R^Q = \frac{\mathbb{C}[x_1, \ldots, x_5]}{\partial_1 W(x) \cdot \ldots \cdot \partial_5 W(x)}.$$

and a Riemanian flat metric $h_{\mu\nu}(t)$. The metric $h_{\mu\nu}(t=0)$ equal to $\eta_{\mu\nu}$. The structure constants are derivatives of Frobenius potential $F(t)$,

$$C_{\mu\nu\lambda}(t) = C^{\rho}_{\mu\nu}(t) h_{\rho\lambda} = \nabla_{\mu}\nabla_{\nu}\nabla_{\lambda} F(t),$$

where ∇_{μ} is Levi-Civitta connection for $h_{\mu\nu}(t)$.
The Frobenius potential $F(t)$ coincides with the prepotential of the Special geometry after restricting on the subspace of the CY complex structure deformations .
$C_{\mu\nu\lambda}(t)$ are nothing but Yukawa coupling constants.

Rezume,

For a generic invariant deformations $W(x) = W_0(x) + \sum t^\mu e_\mu(x) = 0$ does not define a surface in a projective space.

This only occurs when $W(x)$ is quasihomogeneous, i.e. in a case of marginal deformations that is deformations that have the same scaling property as $W_0(x)$.

We denote the marginal deformation parameters $\{\phi^s\} \subset \{t^\alpha\}$.

The marginal deformations $W_0(x) + \sum \phi^s e_s(x)$ define a subspace of the Special Frobenius manifold connected with W_0.

And this subspace coincides with the moduli space of the CY manifold.

The idea of computation of the periods.

Now we can relate the oscilatory form of the period integrals with FM structure and with FM metric $\eta_{\mu\nu}$ in particular. Consider differentials

$$D^{\pm} = D^{\pm}_{W_0} = \mathrm{d} \pm \mathrm{d}W_0 \wedge .$$

They define cohomology subgroups $H^5_{D^{\pm}}(\mathbb{C}^5)_{w \in d \cdot \mathbb{Z}}$ on the space of differential forms of the weght $d \cdot \mathbb{Z}$. They are isomorphic to the Special ring R^Q

$$e_\mu(x) \to e_\mu(x)\mathrm{d}^5x.$$

Cohomology subgroups $H^5_{D^{\pm}}(\mathbb{C}^5)_{w \in d \cdot \mathbb{Z}}$ are dual to the homology subgroups $H_5(\mathbb{C}^5, \mathrm{Re}W_0(x) = \mp\infty)_{w \in d \cdot \mathbb{Z}}$ that consist of cycles $\Gamma^{\pm}_{\mu} \subset H_5(\mathbb{C}^5,\ \mathrm{Re}W_0(x) = \pm\infty)$ with non degenerate pairing with $e_\nu(x)\mathrm{d}^5x \in H^5_{D^{\pm}}(\mathbb{C}^5)_{w \in d \cdot \mathbb{Z}}$ defined as

$$\langle \Gamma^{\pm}_{\mu},\ e_\nu \mathrm{d}^5x \rangle = \int_{\Gamma^{\pm}_{\mu}} e_\nu \cdot e^{\mp W_0(x)} \mathrm{d}^5x.$$

$H_5(\mathbb{C}^5,\ \mathrm{Re}W_0(x) = \pm\infty)_{w \in d \cdot \mathbb{Z}}$ are invariant if $x_i \to e^{2\pi i k_i/d} x_i$.

A possible choice of cycles $\Gamma^{\pm}_{\mu}$ is

$$\int_{\Gamma^{\pm}_{\mu}} e_{\nu} \cdot e^{\mp W_0(x)} \mathrm{d}^5 x = \delta^{\mu}_{\nu}.$$

To compute the periods presented as the oscilatory integrals

$$\int_{\Gamma^{\pm}_{\mu}} e_{\nu} \cdot e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

$W(x,\phi) = W_0(x) + \sum_{s=0}^{\mu} \phi^s e_s(x)$ we first expand the integrand to series over ϕ^s we get the integrals of type $\int_{\Gamma^{\pm}_{\mu}} P(x) e^{-W_0(x)} \mathrm{d}^5 x$ where $P(x) \in R^Q$ being the products $e_s(x)$.
Computing such integrals then is based on the fact that

$$\int_{\Gamma^{\pm}_{\mu}} P(x) e^{\mp W_0(x)} \mathrm{d}^5 x = \int_{\Gamma^{\pm}_{\mu}} \tilde{P}(x) e^{\mp W_0(x)} \mathrm{d}^5 x$$

if the forms in integrands are equivalient in $H^5_{D^{\pm}}(\mathbb{C}^5)_{w \in d \cdot \mathbb{Z}}$

$$P(x)\mathrm{d}^5 x - \tilde{P}(x)\mathrm{d}^5 x = D^{\pm} U.$$

This reduces computing the integrals to the linear problem of expanding $P(x)\mathrm{d}^5 x$ over basis of $H^5_{D^{\pm}}(\mathbb{C}^5)_{w \in d \cdot \mathbb{Z}}$.

The connection between $\eta_{\alpha\beta}$ and $C^{\mu\nu}$.

We use the connection of the CY moduli space to the Special FM to find the inverse intersection matrix of the cycles $C^{\mu\nu}$, $q_\mu \cap q_\nu = Q^+_\mu \cap Q^-_\nu$.
To do this, introduce a few additional basises of periods $\omega^\pm_{\alpha,\mu}(\phi)$ as integrals of $e_\alpha(x)\mathrm{d}^5x \in H^5_{D^\pm}(\mathbb{C}^5)_{w\in d\cdot\mathbb{Z}}$ over the cycles $Q^\pm_\mu \in H_5(\mathbb{C}^5,\ \mathrm{Re}W_0(x) = \pm\infty)_{w\in d\cdot\mathbb{Z}}$ that have been defined earlier:

$$\omega^\pm_{\alpha\mu}(\phi) = \int_{Q^\pm_\mu} e_\alpha(x)\, e^{\mp W(x,\phi)}\mathrm{d}^5x.$$

In particular, the periods $\omega^\pm_{1\mu}(\phi)$ coincide with the periods $\omega^\pm_\mu(\phi)$ defined above since we assume that $e_1(x) = 1$ denotes the unity in the ring R^Q.
The crucial fact for possibility to compute $C^{\mu\nu}$ is its connection with the FM metric $h_{\alpha\beta}(t = 0)$ as:

$$\eta_{\alpha\beta} = \omega^+_{\alpha,\mu}(t = 0)C^{\mu\nu}\omega^-_{\beta,\nu}(t = 0)$$

Proving the connection between $\eta_{\alpha\beta}$ and $C^{\mu\nu}$.

So we need to prove the relation

$$\eta_{ab} = h_{ab}(t=0) = \mathrm{Res}\frac{e_a \cdot e_b \,\mathrm{d}^n x}{\partial_1 W_0 \cdots \partial_n W_0} =$$
$$= \int_{Q^+_\mu} e_a \; e^{-W_0}\mathrm{d}^n x \; C^{\mu\nu} \int_{Q^-_\nu} e_b \; e^{W_0}\mathrm{d}^n x$$

To do this consider a small perturbation $W(x,t) = W_0(x) + t_a e_a$, so that 0 - critical point of W becomes a set of Morse points $p_1, \ldots, p_\mu$ and consider a bilinear form

$$h_{ab}(t,z) = \int_{Q^+_\mu} e_a \; e^{-W(x,t)/z}\mathrm{d}^n x \; C^{\mu\nu} \int_{Q^-_\nu} e_b \; e^{W(x,t)/z}\mathrm{d}^n x$$

Notice, that

$$h_{ab}(t=0,z) = z^k \cdot h_{ab}(t=0, z=1),$$

because if $t=0$, we can absorb z by coordinate transform $x_i \to z^{k_i/d} x_i$.

We can choose basis of cycles : $L_i^{\pm}$ to start from p_i and go along the gradient of $\mathrm{Re}(W(x,t))$ in positive/negative direction and their intersections $L_i^+ \cap L_j^- = \delta_{ij}$. In this basis rhs becomes:

$$\sum_{i=1}^{\mu} \int_{L_i^+} e_a \; e^{-W(x,t)/z} \mathrm{d}^n x \; \int_{L_i^-} e_b \; e^{W(x,t)/z} \mathrm{d}^n x$$

Using stationary phase expansion as $z \to 0$ we obtain for a period:

$$\int_{L_i^+} e_a(x) \; e^{-W(x,t)/z} \mathrm{d}^n x = \pm \frac{(2\pi z)^{N/2}}{\sqrt{\mathrm{Hess}\, W(p_i,t)}} (e_a(p_i) + O(z))$$

From this we get

$$h_{ab}(t,z) = \pm \sum_{i=1}^{\mu} (2\pi i z)^N \frac{e_a(p_i) \cdot e_b(p_i)}{\mathrm{Hess}(W(p_i,t))} (1 + O(z)) =$$

$$= (2\pi i z)^N \left(\mathrm{Res} \frac{e_a \cdot e_b \mathrm{d}^n x}{\partial_1 W \cdots \partial_N W} + O(z) \right)$$

By analytic continuation it holds for $t = 0$. Also we have $h_{ab}(0,z) = z^k \cdot h_{ab}(0,1)$. The above equality now follows from the previous formula.

Finding $C^{\mu\nu}$ and Kahler potential

From this formula we can obtain the expression for $C^{\mu\nu}$ if we know values of $\omega^{+}_{\alpha,\mu}(t=0)$ for all α.
It follows from the their definition

$$\omega^{\pm}_{\alpha\mu}(\phi) = \int_{Q^{\pm}_{\mu}} e_{\alpha}(x)\, e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

that we can express $\omega^{+}_{\alpha,\mu}(t=0)$ in terms of a few first derivatives over ϕ of the periods $\omega^{\pm}_{\mu}(\phi)$ for $\phi = 0$. Denote

$$\omega^{\pm}_{\alpha,\mu}(\phi = 0) := (T^{\pm})^{\alpha}_{\mu}.$$

inserting this to the eq-n above we obtain the relation

$$\eta^{\mu\nu} = (T^{+})^{\mu}_{\rho}\; C^{\rho\sigma}\; (T^{-})^{\nu}_{\sigma}$$

which helps to express intersection matrix $C^{\rho\sigma}$ in terms $\eta^{\mu\nu}$ and matrix T.The result we can insert to the Kahler potential formula

$$e^{-K(\phi)} = \omega_{\mu}(\phi) C^{\mu\nu} \bar{\omega}_{\nu}(\phi)$$

to obtain the explicit expression for $K(\phi)$.

The second basis of cycles

To get more convenient expression for $K(\phi)$ we define one more basis of periods $\sigma^{\pm}_{\mu}(\phi)$ as integrals over the cycles $\Gamma^{\pm}_{\mu} \in H_5(\mathbb{C}^5,\ \mathrm{Re} W_0(x) = \pm\infty)_{w \in d\cdot\mathbb{Z}}$ defined above:

$$\sigma^{\pm}_{\mu}(\phi) = \int_{\Gamma^{\pm}_{\mu}} e^{\mp W(x,\phi)} \mathrm{d}^5 x,$$

Once we have an oscillatory representation for the periods $\sigma^{\pm}_{\mu}(\phi)$ over the corresponding cycles $\Gamma^{\pm}_{\mu}$, we can define additional integrals $\sigma^{\pm}_{\alpha,\mu}(\phi)$ over the same cycles as

$$\sigma^{\pm}_{\alpha,\mu}(\phi) = \int_{\Gamma^{\pm}_{\mu}} e_{\alpha}(x)\, e^{\mp W(x,\phi)} \mathrm{d}^5 x$$

It follows from $e_1(x) = 1$ that $\sigma^{\pm}_{1\mu} = \sigma^{\pm}_{\mu}$. Due to our choice of the cycles $\Gamma^{\pm}_{\mu}$ we also have $\sigma^{\pm}_{\alpha,\mu}(t=0) = \delta_{\alpha,\mu}$.

The connection between two basises of periods

Since both $\omega^{\pm}_{\mu}(\phi)$ and $\sigma^{\pm}_{\nu}(\phi)$ are basises of periods defined as the integrals over the cycles in $H_5(\mathbb{C}^5,\ \mathrm{Re}W_0(x) = \pm\infty)_{w\in d\cdot\mathbb{Z}}$, they are connected by some constant matrix $(T^{\pm})^{\nu}_{\mu}$:

$$\omega^{\pm}_{\mu}(\phi) = (T^{\pm})^{\nu}_{\mu}\ \sigma^{\pm}_{\nu}(\phi).$$

To find T, it suffices to take a few first terms of the expansion over ϕ of the periods $\omega^{\pm}_{\mu}(\phi)$ and $\sigma^{\pm}_{\mu}(\phi)$. The same relation connects periods $\omega^{\pm}_{\alpha\mu}(\phi)$ and $\sigma^{\pm}_{\alpha\nu}(\phi)$ for each α.
Knowing that $\sigma^{\pm}_{\alpha,\mu}(\phi = 0) = \delta_{\alpha,\mu}$, we obtain

$$\omega^{\pm}_{\alpha,\mu}(\phi = 0) = (T^{\pm})^{\alpha}_{\mu}.$$

From above eq-n we then obtain

$$\eta^{\mu\nu} = (T^{+})^{\mu}_{\rho}\ C^{\rho\sigma}\ (T^{-})^{\nu}_{\sigma}.$$

So we express the intersection matrix $C^{\rho\sigma}$ in terms of the known Frobenius metric $\eta^{\mu\nu}$ and the also known matrix T.

Main statement.

Thus we arrive to the main statement that

$$e^{-K(\phi)} = \sigma_\mu(\phi)\, \eta^{\mu\nu}\ M_\nu^\lambda\ \overline{\sigma_\lambda^-, (\phi)}$$

where the matrix $M_b^a = (T^{-1})_c^a\ \bar{T}_b^c$.

It gives an explicit expression for the Kähler potential K in terms of the periods $\sigma_\mu(\phi)$, FM metric $\eta_{\mu\nu}$ and matrix T_ν^μ.
All these data can be computed exactly as it has been explained above.
It makes sense to stress that having the exact expression for $\omega_\nu^\pm(\phi)$, we can obtain the exact and explicit expressions for the periods $\sigma_\mu^\pm(\phi)$:

$$\sigma_\mu^\pm(\phi) = \left((T^\pm)^{-1}\right)_\mu^\nu\ \omega_\nu^\pm(\phi).$$

In terms of the periods $\sigma_\mu^\pm(\phi)$ expression for the Kähler potential has a convenient form for calculating the metric on the CY moduli space.

Example 1: Quintic

The one-parameter family of CY manifold is defined as

$$X_\psi = \{x_i \in \mathbb{P}^4 \mid W_\psi(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - \\ - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\}.$$

In this case, the phase symmetry is $\mathbb{Z}_5^5$ and the induced action $\mathcal{A}$ on the one-dimensional space $\{\psi\}$ is $\mathbb{Z}_5 : \psi \to e^{2\pi i/5}\psi$.
That is the whole complex structure moduli space of the quotient $X/\mathbb{Z}_5^3 =: \hat{X}$, that is the mirror manifold of the original quintic. In particular, $h_{1,1}(\hat{X}) = 101,\ h_{2,1}(\hat{X}) = 1$.
We choose cycles $\Gamma_\mu^\pm$ dual to the cohomology classes $\mathrm{d}^5 x,\ \prod x_i \cdot \mathrm{d}^5 x,\ \prod x_i^2 \cdot \mathrm{d}^5 x,\ \prod x_i^3 \cdot \mathrm{d}^5 x$, a basis in the cohomology subgroup invariant under the $\mathbb{Z}_5^3$.
For the periods, the recursion procedure gives:

$$\sigma_\mu^\pm(\psi) = \frac{(\pm 1)^{\mu-1}}{\Gamma(\mu/5)^5 5^\mu \psi} \sum_{n=0}^{\infty} \frac{\Gamma^5(n+\mu/5)}{\Gamma(5n+\mu)} (5\psi)^{5n+\mu} = \\ = \frac{(\pm\psi)^{\mu-1}}{\Gamma(\mu)} + O(\psi^{\mu+3})$$

The fundamental period for the quintic is defined as a residue of a holomorphic three-form Ω

$$\frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial P_\psi / \partial x_4},$$

and given by an integral over a cycle q_1, which is three-dimensional torus. Its analytic continuations as explained give the whole basis of periods in a basis of cycles with integral coefficients:

$$\omega_\mu(\psi) = \sum_{m=1}^{\infty} \frac{e^{4\pi i m/5}\Gamma(m/5)(5e^{2\pi i(\mu-1)/5}\psi)^{m-1}}{\Gamma(m)\Gamma^4(1-m/5)}, \quad |\psi| < 1,$$

Taking the first four terms of the expansion of the periods above we obtain

$$T^\mu_\nu = \frac{5^{\nu-1} e^{2\pi i((\nu-1)(\mu-1)+2\nu)/5}\Gamma(\nu/5)}{\Gamma^4(1-\nu/5)},$$

The FM holomorphic metric in this case

$$\eta = \mathrm{antidiag}(1,1,1,1).$$

Finally we obtain $\hat{\eta} = \eta T^{-1}\bar{T}$ and Kähler potential for the metric:

$$e^{-K(\psi)} = \frac{\Gamma^5(1/5)}{125\Gamma^5(4/5)}\sigma^+_{11}\overline{\sigma^-_{11}} + \frac{\Gamma^5(2/5)}{5\Gamma^5(3/5)}\sigma^+_{12}\overline{\sigma^-_{12}} + \\ + \frac{5\Gamma^5(3/5)}{\Gamma^5(2/5)}\sigma^+_{13}\overline{\sigma^-_{13}} + \frac{125\Gamma^5(4/5)}{\Gamma^5(1/5)}\sigma^+_{14}\overline{\sigma^-_{14}}.$$

In particular,

$$G_{\psi\overline{\psi}}(0) = 25\frac{\Gamma^5(4/5)\Gamma^5(2/5)}{\Gamma^5(1/5)\Gamma^5(3/5)}$$

that coincides with the famous result by Candelas et al.

Example 2: Fermat hypersurface

The direct generalization of the quintic is a Fermat hypersurface, which is the one given by the equation

$$W_0(x) = \sum_{i=1}^{5} x_i^{n_i}, \qquad n_i = d/k_i, \quad \sum k_i = d,$$

and the degree d is equal to the least common multiple of $\{k_i\}$. As in the case above, we consider a one-dimensional deformation $W(x, \phi_0) = W_0(x) + \phi_0 \prod_{i=1}^{5} x_i$. The phase symmetry group is $\Pi_X = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_5}$. The lifted action on ϕ_0 is $\mathbb{Z}_d: \ \phi_0 \to \zeta\phi_0, \ \zeta = e^{2\pi i/d}$. We take the expression for the fundamental period the known result by Berglund et al:

$$\omega_1(\phi_0) == \sum_{\mu=1}^{d-1} A(\mu) \frac{\phi_0^{\mu-1}}{\Gamma(\mu)} + O(\phi_0^{d-1}).$$

and

$$A(\mu) = \frac{(-1)^{\mu-1} e^{\frac{-\pi i \mu}{d}}}{\sin \frac{\mu\pi}{d} \prod_{i=1}^{5} \Gamma(1 - \frac{k_i \mu}{d})}$$

We note that $A(\mu)$ vanishes if $k_i\mu/d \in \mathbb{Z}$, i.e. $\mu/n_i \in \mathbb{Z}$. According to the general analytic continuation procedure

$$\omega_\mu(\phi_0) = \sum \zeta^{(\nu-1)(\mu-1)} A(\nu) \frac{\phi_0^{\nu-1}}{\Gamma(\nu)} + O(\phi_0^{d-1}).$$

Using the definitions for $\sigma_\mu^+(\phi_0)$ we obtain

$$\sigma_\mu^+(\phi_0) = \frac{\phi_0^{\mu-1}}{\Gamma(\mu)} + O(\phi_0^{\mu+d-2}), \qquad \mu/n_i \notin \mathbb{Z}, \text{ otherwise } 0$$

This latter condition implies that ω_μ form a basis in the periods of Ω deformed by ϕ_0. We obtain the transition matrix

$$T_\nu^\mu = \zeta^{(\mu-1)(\nu-1)} A(\mu), \qquad \mu/n_i \notin \mathbb{Z}, \qquad \nu/n_i \notin \mathbb{Z}$$

$$(T^{-1})_\mu^\lambda = \frac{\bar{\zeta}^{(\lambda-1)(\nu-1)}}{\tilde{d}-1} \frac{1}{A(\mu)}$$

and the real structure

$$M_\nu^\mu = \frac{\bar{A}(\mu)}{A(d-\mu)} \delta_{\mu+\nu,d} \ .$$

In this case, $\eta_{\mu,\nu} = \delta_{\mu+\nu,d}$ and therefore

$$e^{-K(\phi_0)} = \sum_{\mu=1,\ \mu/n_i \notin \mathbb{Z}}^{d-1} \prod_{i=1}^{5} \gamma\left(\frac{k_i \mu}{d}\right) \sigma_\mu^+(\phi_0)\overline{\sigma_\mu^-(\phi_0)}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and

$$\sigma_\mu^\pm(\phi_0) = \pm \sum_{R=0}^{\infty} \frac{\phi_0^{\mu-1+dR}}{\Gamma(dR+\mu)} \prod_{j=1}^{5} \frac{\Gamma\left(k_j(R+\frac{\mu}{d})\right)}{\Gamma(\frac{k_j \mu}{d})}$$

From this we get a formula for the metric itself

$$G_{\phi_0\overline{\phi_0}} = \prod_{i=1}^{5}\left(\gamma\left(\frac{k_i\mu_0}{d}\right)\gamma\left(1-\frac{k_i}{d}\right)\right)\frac{|\phi_0|^{2(\mu_0-1)}}{\Gamma(\mu_0)^2} + O(|\phi_0|^{2\mu_0}),$$

μ_0 is the least integer $1 \leq \mu_0 < d$ such that $(\mu_0+1)/n_j \neq \mathbb{Z}$.
The last formula reproduces the known results for CY manifolds $\mathbb{P}^4_{(2,1,1,1,1)}[6]$, $\mathbb{P}^4_{(4,1,1,1,1)}[8]$ and $\mathbb{P}^4_{(5,2,1,1,1)}[10]$ obtained by Klemm and Theisen.

Example 3: The case of 5 polynomials

We assume that the above approach is applicable to the case of CY manifold defined in terms of the hypersurface in weighted projective spaces defining polynomial is

$$W_0(x) = \sum_{j=1}^{5} \prod_{i=1}^{5} x_i^{a_{ij}}, \qquad \sum k_i a_{ij} = d,$$

and

$$\sum k_i = d.$$

In this case periods are given in terms of the *mirror* CY manifold $\hat{X}$. The polynimial $W_0(x)$ has a group Π_X of phase symmetries represented as

$$\Pi_X = Q_X \times G_X,$$

where Q_x, a *quantum symmetry* group $\simeq (\mathbb{Z}_d : k_1, \cdots, k_5)$, acts as $x_i \to e^{2\pi i k_i/d}$. We note that action of the quantum symmetries on X is trivial. The complement to Q_X in Π_X is called a geometric symmetry group G_X.

For mirror manifolds the total phase symmetry is unchanged whereas roles of quantum and geometric symmetries switch:

$$G_X = Q_{\hat{X}}, \qquad Q_X = G_{\hat{X}}.$$

To build such a mirror, we must first to consider a polynomial $\hat{W}_0(x)$ with a transposed matrix of exponents $\hat{a}_{ij} = a_{ji}$,

$$\hat{W}_0(x) = \sum_{j=1}^{5} \prod_{i=1}^{5} x_i^{\hat{a}_{ij}}, \qquad \sum \hat{k}_i a_{ji} = \hat{d},$$

and

$$\sum \hat{k}_i = \hat{d}.$$

Here $\hat{k}_i$ and $\hat{d}$ are uniquely defined by the reqirement that the equalities above are satisfied.

This polynomial has the same group of phase symmetries, however generically the needed condition is not fulfiled, i.e. its quantum symmetry is smaller, than geometric symmetry of the original hypersurface.

To get a mirror we need to enlarge quantum symmetry of $\{\hat{W}_0(x) = 0\}$. For this purpose we take a quotient of the hypersurface $\{\hat{W}_0(x) = 0\}/H$, where H is some subgroup of phase symmetries which is to be found in each case.
Thus, computing complex moduli space for the manifold X (or $\hat{X}$) we compute also a complexified Kähler moduli space metric for the mirror CY through the mirror map.
The periods $\omega_\mu(\phi)$ in this case were computed earlier and, if we set all parameters ϕ^s (but not ϕ_0) equal to zero for simplicity , then we have:

$$\omega_1(\phi_0) = \sum_{r=1}^{\hat{d}-1} A(r) \frac{\phi_0^{r-1}}{\Gamma(r)} + O(\phi_0^{\hat{d}-1})$$

$$A(\mu) = (-1)^\mu \frac{\pi}{\hat{d} \sin \frac{\pi\mu}{\hat{d}}} \prod_{j=1}^{5} \frac{1}{\Gamma(1 - \frac{\hat{k}_j \mu}{\hat{d}})}.$$

For our general method to work, this must give all relevant periods. Basically we must check that all possible periods are obtained from this one (with all $\phi^s \neq 0$) by phase-symmetry analytic continuations.

In other words it is necessary to verify the relation

$$\dim\langle\omega_0(\mathcal{A}(g)\cdot\phi)\rangle_{g\in G_X} = \dim H_3(X).$$

This was certainly the case in the preceding examples, but not in this case, we are not aware of this fact in general (it is so in all examples). As in the previous example, in the one-modulus case we obtain

$$e^{-K(\phi_0)} = \sum_{\mu=1,\ \mu\hat{k}_i/\hat{d}\notin\mathbb{Z}}^{\hat{d}-1} \eta^{\mu,\hat{d}-\nu} \prod_{j=1}^{5} \gamma\left(\frac{\hat{k}_j\mu}{\hat{d}}\right) \sigma_\mu^+(\phi_0)\overline{\sigma_\nu^-(\phi_0)}.$$

For this formula to hold the number of linearly independent elements $\prod_{i=1}^5 x_i^n \mathrm{d}^5 x \in H^5_{D^\pm}(\mathbb{C}^5)$ should be equal to the number of $1 \le \mu < \hat{d},\ \mu k_i/d \neq \mathbb{Z}$.

Example 4: Two-moduli non-Fermat threefold.

This CY manifold is built from the following hypersurface

$$X = \{x_i \in \mathbb{P}^4_{(3,2,2,7,7)} \mid W_\phi(x) = x_0^7 + x_1^7 x_3 + x_3^3 + x_2^7 x_4 + x_4^3 - \\ - \phi_0 x_1 x_2 x_3 x_4 x_5 + \phi_1 x_0^3 x_1^3 x_2^3 = 0\}.$$

This example considered in BCOFHJQ is interesting because it is not of Fermat type and is not described by a product of N=2 Minimal Models. Its mirror is a hypersurface of degree 7 in a different projective space $\mathbb{P}^4_{(1,1,1,2,2)}$.
The weight of the singularity is equal to $d = 21$. The phase symmetry is $\mathbb{Z}_{21}^2 \times \mathbb{Z}^7$. We again consider a factor $\hat{X} = X/H$ by the following $H = \mathbb{Z}_{21}$ action:

$$H := (\mathbb{Z}_{21} \ : \ 12, 2, 0, 7, 0)$$

The Hodge numbers are $h_{1,1}(\hat{X}) = 95, \ h_{2,1}(\hat{X}) = 2$. The above two–parametric family is the maximal deformation surviving after the factorisation. The induced action $\mathcal{A}$ on the two-dimensional space $\{\phi_0, \phi_1\}$ is $\mathbb{Z}_7 : \phi_0 \to \alpha\phi_0, \ \phi_1 \to \alpha^3\phi_1$, where $\alpha^7 = 1$ is a primitive root.

Analytic continuations of the fundamental period give the full basis of periods in a basis of cycles with integral coefficients:

$$\omega_\mu(\phi_0,\phi_1) = -\frac{1}{7}\sum_{n=1}^{\infty} e^{6\pi i n/7}\frac{(\alpha^{\mu-1}\phi_0)^{n-1}}{\Gamma(n)}\sum_{m=0}^{\infty}\frac{e^{-3i\pi m/7}\Gamma\left(\frac{n+3m}{7}\right)}{\Gamma^2\left(1-\frac{n+3m}{7}\right)\Gamma^2\left(1-\frac{2n-}{7}\right.}$$

Now we perform the Milnor ring computations to compute the metric η. If we denote

$$e_2(x) = x_0x_1x_2x_3x_4,\ e_3(x) = x_0^3x_1^3x_2^3,$$

then the H-invariant subring of R_0^Q is generated by e_2 and e_3. It is easy to compute the following relations:

$$e_3^2 = 0,\ e_2^3 = 0$$

and thus the vector space basis of this subring is:

$$e_1,\ e_2,\ e_3,\ e_4 = e_2^2,\ e_5 = e_2e_3,\ e_6 = e_2^2e_3.$$

The last one is of the highest degree 63 and therefore in this basis the metric $\eta = \mathrm{antidiag}(1,1,1,1,1,1)$.

Taking the first four terms of the expansion of the periods above we obtain

$$T_\mu^\nu = A(\nu)\alpha^{k_\nu(\mu-1)}, \; k_\nu = (1,2,4,3,5,6)$$

$$A(\nu) = \alpha^{2m_\nu - n_\nu/2} \frac{(-1)^{n_\nu - 1}\Gamma\left(\frac{n_\nu+3m_\nu}{7}\right)}{\Gamma^2\left(1-\frac{n_\nu+3m_\nu}{7}\right)\Gamma^2\left(1-\frac{2n_\nu-m_\nu}{7}\right)},$$

Here $(n_\nu, m_\nu) = ((1,0),(2,0),(1,1),(3,0),(2,1),(3,1))$ correspond to our choice of basis.

The Kähler potential for the metric:

$$e^{-K(\phi_0,\phi_1)} = \gamma^3(1/7)\gamma^2(2/7)\sigma_{11}^+\overline{\sigma_{11}^-} + \gamma^3(2/7)\gamma^2(4/7)\sigma_{12}^+\overline{\sigma_{12}^-} + \gamma^3(4/7)\gamma^2$$
$$\gamma^3(3/7)\gamma^2(6/7)\sigma_{14}^+\overline{\sigma_{14}^-} + \gamma^3(5/7)\gamma^2(3/7)\sigma_{15}^+\overline{\sigma_{15}^-} + \gamma^3(6/7)\gamma^2(5/7)\sigma_{16}^+\overline{\sigma_{16}^-},$$

here $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Kähler metric has the form

$$G(0) = \begin{pmatrix} \gamma^3\left(\frac{6}{7}\right)\gamma^2\left(\frac{4}{7}\right)\gamma\left(\frac{2}{7}\right) & 0 \\ 0 & \gamma^3\left(\frac{4}{7}\right)\gamma^2\left(\frac{5}{7}\right)\gamma\left(\frac{6}{7}\right) \end{pmatrix}$$

Conclusion

A new method for computing the metric of CY moduli space is proposed. This method does not demand using of Picard–Fuchs equations. Instead, the cohomology technique for computing periods can be applied. It can be used for the computations of the CY moduli space geometry in cases when the dimesion of the moduli space more than one.

The Special FM structure naturally arising from an N=2 SCFT plays a significant role. The result is given in terms of the topological metric on FM and two basises of periods.
Both of these basises can be computed avoiding the complicated direct computation of the symplectic basis of periods.

The method used here was applied for CY manifolds, given by one polynomial equation, such as the case of Fermat hypersurfaces. We suppose the same approach can be used for CY manifolds of a more general type.