Palatini cosmology in different frames

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Einstein gravity is a beautiful theory which is very well tested in the Solar system scale. However it indicates some drawbacks in the other scales. The simplest way to generalize (/modify) it is by replacing Einstein-Hilbert Lagrangian

$$R \to f(R) = R - 2\Lambda + \gamma R^2 + \cdots = \sum_{i=0}^{n} \gamma_i R^i$$

by an arbitrary function of the scalar R. Such modification might be helpful in solving dark matter and dark energy problems .

Here we focus on some cosmological applications presented in
arXiv:1707.01948; Eur.Phys.J. C77 (2017) no.9, 603
arXiv:1608.03196; Eur.Phys.J. C77 (2017) no.6, 406
arXiv:1512.04580; Eur.Phys.J. C76 (2016) no.10, 567,
arXiv:1512.01199; JCAP01 (2016) 040

Introduction I

In the Palatini $f(\hat{R})$ gravity the action is dependent on a metric and a torsionless connection as independent variables

$$S(g_{\mu\nu},\Gamma^{\lambda}_{\rho\sigma}) = S_{\mathrm{g}} + S_{\mathrm{m}} = \frac{1}{2}\int \sqrt{-g} f(\hat{R}) d^4x + S_{\mathrm{m}}(g_{\mu\nu},\psi), \quad (1)$$

where $\hat{R}(g,\Gamma) = g^{\mu\nu}\hat{R}_{\mu\nu}(\Gamma)$ is the generalized Ricci scalar and $\hat{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ. EOM are

$$f'(\hat{R})\hat{R}_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(\hat{R})g_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

$$\hat{\nabla}_{\alpha}(\sqrt{-g}f'(\hat{R})g^{\mu\nu}) = 0, \quad (3)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L_{\mathrm{m}}}{\delta g^{\mu\nu}}$ (e.g. $\mathrm{PF} = (p+\rho)u_{\mu}u_{\nu} + pg_{\mu\nu}$) is EMT, i.e. assuming that the matter couples minimally to the metric $g_{\mu\nu}$.

Introduction II

In order to solve equation (3) it is convenient to introduce a new metric

$$\sqrt{-\bar{g}}\bar{g}_{\mu\nu} = \sqrt{-g}f'(\hat{R})g_{\mu\nu} \tag{4}$$

for which the connection $\Gamma = \Gamma_{L-C}(\bar{g})$ is a Levi-Civita connection. As a consequence in dim $M = 4$ one gets

$$\bar{g}_{\mu\nu} = f'(\hat{R})g_{\mu\nu}, \tag{5}$$

For this reason one should assume that the conformal factor $f'(\hat{R}) \neq 0$, so it has strictly positive or negative values. Taking the $g-$trace of (2), we obtain structural equation

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = T. \tag{6}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ $(= 3p - \rho)$. Thus, the equation (2) can be treated both as determining the dynamics of the metric g or $\bar{g}$ (two frames !!)

The eq. (2) can be recast to the following form

$$\bar{R}_{\mu\nu} - \frac{1}{4}\bar{R}\,\bar{g}_{\mu\nu} = \frac{1}{f'(\hat{R})}(T_{\mu\nu} - \frac{1}{4}T\,g_{\mu\nu}). \tag{7}$$

where $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}, \bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} = f'(\hat{R})^{-1}\hat{R}$ and $\bar{g}_{\mu\nu}\bar{R} = \ g_{\mu\nu}\hat{R}$.
1. non-linear system of second order PDE.
2. for the linear Lagrangian $\hat{R} - 2\Lambda$ is fully equivalent to Einstein $R - 2\Lambda$,
3. any $f(\hat{R})$ vacuum solutions ($T_{\mu\nu} = 0$) $\Rightarrow$ Einstein vacuum solutions with cosmological constant;
4. PF: $T_{\mu\nu} = (p+\rho)u_\mu u_\nu + pg_{\mu\nu} \Rightarrow$
$T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu} = (p+\rho)\left(u_\mu u_\nu + \frac{1}{4}g_{\mu\nu}\right)$. Thus DE solutions $\equiv$ vacuum solutions.

Palatini gravity is the first cousin of Einstein theory (next of kin)!!

The action (1) is dynamically equivalent to the constraint system with first order Palatini gravitational Lagrangian with the additional scalar field χ, provided that $f''(\hat{R}) \neq 0$ (This condition excludes the linear Einstein-Hilbert Lagrangian $f(\hat{R}) = \hat{R} - 2\Lambda$ from our considerations.)

$$S(\chi, g_{\mu\nu}, \Gamma^{\lambda}_{\rho\sigma}) = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(f'(\chi)(\hat{R}-\chi) + f(\chi)\right) + S_m(g_{\mu\nu}, \psi), \tag{8}$$

Introducing new scalar field $\Phi = f'(\chi)$ and taking into account the constraint equation $\chi = \hat{R}$, one can rewrite the action in dynamically equivalent way as a Palatini action

$$S(\Phi, g_{\mu\nu}, \Gamma^{\lambda}_{\rho\sigma}) = \frac{1}{2k}\int \mathrm{d}^4x\sqrt{-g}\left(\Phi\hat{R} - U(\Phi)\right) + S_m(g_{\mu\nu}, \psi), \tag{9}$$

where the potential $U(\Phi)$ encodes the information about the function $f(\hat{R})$ is given by

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \tag{10}$$

and $\Phi = \frac{df(\chi)}{d\chi}$. Thus one has $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$. For a given f the potential U is a (singular) solution of the Clairaut's differential equation: $U(\Phi) = \Phi\frac{dU}{d\Phi} - f(\frac{dU}{d\Phi})$. (One can observe that the trivial, i.e. constant, potential $U(\Phi)$ corresponds to the linear Lagrangian $f(\hat{R}) = \hat{R} - 2\Lambda$.) Palatini variation of this action provides

$$\Phi\left(\hat{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\hat{R}\right) + \frac{1}{2}g_{\mu\nu}U(\Phi) - \kappa T_{\mu\nu} = 0 \tag{11a}$$

$$\hat{\nabla}_\lambda(\sqrt{-g}\Phi g^{\mu\nu}) = 0 \tag{11b}$$

$$\hat{R} - U'(\Phi) = 0 \tag{11c}$$

The last equation due to the constraint $\hat{R} = \chi = U'(\phi)$ is automatically satisfied. The middle equation (11b) implies that the connection $\hat{\Gamma}$ is a metric connection for the new metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$.

Now the equation (11a), (11c) can be written as a dynamical equation for the metric $\bar{g}_{\mu\nu}$ ($\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}, \hat{R} = \Phi\bar{R}, g_{\mu\nu}\hat{R} = \bar{g}_{\mu\nu}\bar{R}$)

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \kappa\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{U}(\Phi) \tag{12a}$$

$$\Phi\bar{R} - (\Phi^2\,\bar{U}^(\Phi))' = 0 \tag{12b}$$

where we have introduced $\bar{U}(\phi) = U(\phi)/\Phi^2$ and $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$. Thus the system (12a) - (12b) corresponds to a scalar-tensor action for the metric $\bar{g}_{\mu\nu}$ and the (non-dynamical) scalar field Φ

$$S(\bar{g}_{\mu\nu}, \Phi) = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left(\bar{R} - \bar{U}(\Phi)\right) + S_m(\Phi^{-1}\bar{g}_{\mu\nu}, \psi), \tag{13}$$

non-minimally coupled to the matter ψ.

where

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\mu\nu}}S_m = (\bar{\rho}+\bar{p})\bar{u}^\mu\bar{u}^\nu + \bar{p}\bar{g}^{\mu\nu} = \Phi^{-3}T^{\mu\nu} \ , \quad (14)$$

and $\bar{u}^\mu = \Phi^{-\frac{1}{2}}u^\mu$, $\bar{\rho} = \Phi^{-2}\rho$, $\bar{p} = \Phi^{-2}p$, $w = \bar{w}$
$\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$, $\bar{T} = \Phi^{-2}T$. Further, the trace of (12a), provides

$$\bar{R} = 2\bar{U}(\Phi) - \kappa\bar{T} \quad (15)$$

The equation (12a), due to non-minimal coupling between the metric $\bar{g}_{\mu\nu}$ and the matter, implies eneregy-momentum non-conservation

$$\bar{\nabla}^\mu\bar{T}_{\mu\nu} = -\frac{1}{2}\bar{T}\frac{\partial_\nu\Phi}{\Phi} \quad (16)$$

(however $\nabla^\mu T_{\mu\nu} = 0$). In this, so-called Einstein frame case, the scalar field has no dynamics satisfying algebraic equation (12b).

By changing the frame $(\bar{g}_{\mu\nu}, \Phi) \to (g_{\mu\nu}, \Phi)$ one gets that action for the original Palatini metric within scalar-tensor formulation

$$S(\Phi, g_{\mu\nu}) = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(\Phi R + \frac{3}{2\Phi}\partial_\mu\Phi\partial^\mu\Phi - U(\Phi)\right), \quad (17)$$

where $U(\Phi)$ is given as before by (10).

In this case, a kinematical part of the scalar field does not vanish from the Lagrangian (17). We obtain Brans-Dicke action with the parameter $\omega_{BD} = -\frac{3}{2}$ in the Jordan frame. In this case equations of motion take the form ($\nabla^\mu T_{\mu\nu} = 0$)

$$\Phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{3}{4\Phi}g_{\mu\nu}\nabla_\sigma\Phi\nabla^\sigma\Phi + \frac{3}{2\Phi}\nabla_\mu\Phi\nabla_\nu\Phi + g_{\mu\nu}\Box\Phi - \nabla_\mu\nabla_\nu\Phi + \frac{1}{2}g_{\mu\nu}U(\phi) = \kappa T_{\mu\nu}, \quad (18a)$$

$$R - \frac{3}{\Phi}\Box\Phi + \frac{3}{2\Phi^2}\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{2}U'(\Phi) = 0 \quad (18b)$$

Cosmological applications I

Assume that the metric g is a spatially flat FLRW metric

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (19)$$

where $a(t)$ is the scale factor, t is the cosmic time.
As a source of gravity we assume perfect fluid with the energy-momentum tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho(t), p(t), p(t), p(t)), \qquad (20)$$

where $p = w\rho$, $w = const$ is a form of the equation of state ($w = 0$ for dust and $w = 1/3$ for radiation). Formally, effects of the spatial curvature can be also included to the model by introducing a curvature fluid $\rho_{\mathrm{k}} = -\frac{k}{2}a^{-2}$, with the barotropic factor $w = -\frac{1}{3}$ ($p_{\mathrm{k}} = -\frac{1}{3}\rho_{\mathrm{k}}$). From the conservation condition $T^{\mu}_{\nu;\mu} = 0$ we obtain that $\rho = \rho_0 a^{-3(1+w)}$. Therefore, trace T reads as

$$T = \rho_0(3w-1)a(t)^{-3(1+w)}. \qquad (21)$$

Following further Cosmological Principle we assume that Φ depends only on the cosmic time. In such a case the metric $\bar{g}_{\mu\nu} = \Phi(t) g_{\mu\nu}$ is FRW metric as well with a new cosmic time $d\bar{t} = \Phi(t)^{\frac{1}{2}} d\, t$ and new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}} a(\bar{t})$.

$$d\bar{s}^2 = d\bar{t}^2 - \bar{a}^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (22)$$

Similarly

$$\bar{T}^{\mu}_{\nu} = \mathrm{diag}(-\bar{\rho}(\bar{t}), \bar{p}(\bar{t}), \bar{p}(\bar{t}), \bar{p}(\bar{t})), \qquad (23)$$

where $\bar{T}^{\mu}_{\nu} = \Phi^{-2} T_{\mu\nu}$, $\bar{T} = \Phi^{-2} T$, $\bar{\rho} = \Phi^{-2}\rho$, $\bar{p} = \Phi^{-2} p$.
From Einstein equations one gets Friedmann and Raychaudhuri equations

$$3\bar{H}^2 = \bar{\rho}_{\Phi} + \bar{\rho}_m, \quad 3\frac{\ddot{\bar{a}}}{\bar{a}} = \bar{\rho}_{\Phi} - \bar{\rho}_m$$

where $\rho_{\Phi} = \frac{1}{2}\bar{U}(\Phi), \bar{\rho}_m = \rho_0 \bar{a}^{-3}\Phi^{\frac{1}{2}}$ and non-conservation of the matter EMT $\bar{T}_{\mu\nu}$

$$\dot{\bar{\rho}}_m + 3\bar{H}\bar{\rho}_m = -\dot{\bar{\rho}}_{\Phi}$$

is equivalent to the EOM for Φ: $\Phi\bar{R} - (\Phi^2\,\bar{U}^{(}\Phi))' = 0$

We consider visible and dark matter in the form of dust $w = 0$ and choose $f(\hat{R}) = \sum_{i=0}^{n} \gamma_i \hat{R}^i$ including Palatini version of the Starobinski model with a quadratic Ricci scalar term. Assuming spatially flat FLRW cosmology with a dust source one gets Friedmann eq. ($H = \frac{d \ln a}{dt}$) in the Jordan frame

$$H^2 = \frac{2f'(3f - f'\hat{R})}{3\left(2f' + \frac{3(2f - \hat{R}f')(f'')}{f''\hat{R} - f'}\right)^2}, \tag{24}$$

where the prime denotes differentiation with respect to $\hat{R}$. Because the form of a function $f(\hat{R})$ is unknown, one can probe the simplest modification of general relativity Lagrangian

$$f(\hat{R}) = -2\Lambda + \hat{R} + \gamma \hat{R}^2 \cdots (+\delta \hat{R}^3) \tag{25}$$

induced by first three (/four) terms in the power series decomposition of an arbitrary function $f(R)$.

The Lagrangian (25) can be viewed as a simplest deviation, by the quadratic Starobinsky term, from the Lagrangian $\hat{R} - 2\Lambda$ which provides the standard cosmological model a.k.a. ΛCDM model.

It appears that a corresponding solution of the structural equation (6)

$$\hat{R} = 4\Lambda + \rho_{m,0}a^{-3} \equiv 4\rho_{\Lambda,0} + \rho_{m,0}a^{-3} \tag{26}$$

The solution (49) is to be plugged into the formula (24) which generalizes Friedmann equation

$$\frac{H^2}{H_0^2} = \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} \tag{27}$$

for ΛCDM model. A counterpart of the formula above in our extended model can be presented as follows

$$\frac{H^2}{H_0^2} = \frac{b^2}{\left(b+\frac{d}{2}\right)^2} \times \left(\Omega_\gamma(\Omega_{m,0}a^{-3}+\Omega_{\Lambda,0})^2\frac{(K-3)(K+1)}{2b}+\Omega_{m,0}a^{-3}+\Omega_{\Lambda,0}\right),$$

where

$$K = \frac{3\Omega_{\Lambda,0}}{\Omega_{\mathrm{m},0}}a^{-3}+\Omega_{\Lambda,0}, \tag{28}$$

$$\Omega_\gamma = 3\gamma H_0^2, \tag{29}$$

$$b = f'(\hat{R}) = 1+2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3}+4\Omega_{\Lambda,0}), \tag{30}$$

$$d = \frac{1}{H}\frac{db}{dt} = -2\Omega_\gamma(\Omega_{\mathrm{m}}a^{-3}+\Omega_{\Lambda,0})(3-K) \tag{31}$$

The study of this Friedmann equation is a main subject of this talk.

Cosmological dynamical system of Newtonian type I

Consider general form of Friedmann equations

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = F(a) > 0, \tag{32}$$

It would be convenient to rewrite (32) it in equivalent form

$$\frac{1}{2}\dot{a}^2 + V(a) = 0, \tag{33}$$

as a zero energy trajector of the Hamiltonian system $H = \frac{1}{2}\dot{a}^2 + V(a)$, where the potential

$$V(a) = -\frac{1}{2}a^2 F(a) < 0 \tag{34}$$

This implies Newton type equation

$$\ddot{a} = -\frac{\partial V}{\partial a}, \qquad t = \int^a \frac{da}{\sqrt{-2V(a)}} \tag{35}$$

Cosmological dynamical system of Newtonian type II

Accordingly the evolution of a universe can be interpreted, in dual picture, as a motion of a fictitious particle of unit mass in the potential $V(a)$. The corresponding dynamical system in two-dimensional phase space $(a, x = \dot{a})$

$$\dot{a} = x, \tag{36}$$

$$\dot{x} = -\frac{\partial V(a)}{\partial a}. \tag{37}$$

Phase space portrait consists of all possible trajectories corresponding to all possible energy levels

$$\left\{(a, \dot{a})\colon \frac{\dot{a}^2}{2} + V(a) = E; E \in \mathbb{R}\right\}. \tag{38}$$

For example for the standard cosmological model (27)

$$V = -\frac{a^2}{6}\left(\rho_{m,0}a^{-3} + \rho_{\Lambda,0}\right), \tag{39}$$

Cosmological dynamical system of Newtonian type III

In a case of singularitiese on needs theory of piecewise smooth dynamical systems. Therefore it is assumed that the potential function, except some isolated (singular) points, belongs to the class C^2).

Any cosmological model can be identified by its form of the potential function $V(a)$ depending on the scale factor a. From the Newtonian form of the dynamical system (36)-(37) one can see that all critical points correspond to vanishing of r.h.s of the dynamical system $\left(x_0 = 0, \ \frac{\partial V(a)}{\partial a}|_{a=a_0}\right)$. Therefore all critical points are localized on the x-axis, i.e. they represent a static universe.

The only admissible critical points are the saddle type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} < 0$ or centres type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} > 0$.

Cosmological dynamical system of Newtonian type IV

If a form of the potential function is known (from the knowledge of effective energy density), then it is possible to calculate cosmological functions in exact form

$$t = \int^{a} \frac{da}{\sqrt{-2V(a)}}, \tag{40}$$

$$H(a) = a^{-1}\sqrt{-2V(a)}, \tag{41}$$

a deceleration parameter, an effective barotropic factor

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{1}{2}\frac{d\ln(-V)}{d\ln a}, \tag{42}$$

$$w_{\text{eff}}(a(t)) = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{1}{3}\left(\frac{d\ln(-V)}{d\ln a} + 1\right), \tag{43}$$

Cosmological dynamical system of Newtonian type V

a parameter of deviation from de Sitter universe

$$h(t) \equiv -(q(t)+1) = \frac{1}{2}\frac{d\ln(-V)}{d\ln a} \tag{44}$$

(note that if $V(a) = -\frac{\Lambda a^2}{6}$, $h(t) = 0$), effective matter density and pressure

$$\rho_{\text{eff}} = -\frac{6V(a)}{a^2}, \tag{45}$$

$$p_{\text{eff}} = \frac{2V(a)}{a^2}\left(\frac{d\ln(-V)}{d\ln a} + 1\right) \tag{46}$$

and, finally, a Ricci scalar curvature for the FRW metric (**??**)

$$R = \frac{6V(a)}{a^2}\left(\frac{d\ln(-V)}{d\ln a} + 2\right). \tag{47}$$

Polynomial example I

$$\Omega_{\mathrm{R}} = \frac{\hat{R}}{3H_0^2}, \qquad \Omega_{\gamma_i} = 3^{i-1}\gamma_i H_0^{2(i-1)},$$

$$\Omega_{\mathrm{tot}} = \Omega_{\mathrm{m},0}a^{-3} + \Omega_{\Lambda,0}, \qquad b = f'(\hat{R}) = \sum_{i=1}^{n} i\Omega_{\gamma_i}\Omega_R^{i-1},$$

$$d = -3\left(\sum_{i=1}^{n}(i-2)\Omega_{\gamma_i}\Omega_R^{i-1} + \frac{4\Omega_\Lambda}{\Omega_R}\right) \times \frac{\sum_{i=1}^{n} i(i-1)\Omega_{\gamma_i}\Omega_R^{i-1}}{\sum_{i=1}^{n} i(i-2)\Omega_{\gamma_i}\Omega_R^{i-1}}. \quad (48)$$

where H_0 is the present value of Hubble function, $\Omega_{\mathrm{m},0} = \frac{\rho_{\mathrm{m},0}}{3H_0^2}$, $\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{3H_0^2}$.

Polynomial example - Jordan frame I

constraints eq.

$$\sum_{i=1}^{n}(i-2)\Omega_{\gamma_i}\Omega_R^i = -\Omega_{\mathrm{m}} - 4\Omega_\Lambda. \tag{49}$$

Friedmann eq.

$$\frac{H^2}{H_0^2} = \frac{b^2}{\left(b+\frac{d}{2}\right)^2} \times \left[\frac{1}{2b}\left[\sum_{i=1}^{n}\Omega_{\gamma_i}\Omega_R^{i-1}\left(\Omega_R - 2i\Omega_{\mathrm{tot}}\right) + \Omega_{\mathrm{tot}} - 3\Omega_\Lambda\right] + \Omega_{\mathrm{tot}}\right]. \tag{50}$$

Polynomial example - Jordan frame I

$$V(a) = -\frac{H_0^2 a^2}{2} \times \left[\frac{1}{2b}\left[\sum_{i=1}^{n} \Omega_{\gamma_i}\Omega_R^{i-1}\left(\Omega_R - 2i\Omega_{\mathrm{tot}}\right) + \Omega_{\mathrm{tot}} - 3\Omega_\Lambda\right] + \Omega_{\mathrm{tot}}\right]. \quad (51)$$

Polynomial example - Einstein frame

$$S(\bar{g}_{\mu\nu},\Phi)=\frac{1}{2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left(\bar{R}-\bar{U}(\Phi)\right)+S_m(\Phi^{-1}\bar{g}_{\mu\nu},\psi) \quad (52)$$

with non-minimal coupling between Φ and $\bar{g}_{\mu\nu}$

$$\bar{T}^{\mu\nu}=-\frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\mu\nu}}S_m=(\bar{\rho}+\bar{p})\bar{u}^{\mu}\bar{u}^{\nu}+\bar{p}\bar{g}^{\mu\nu}=\Phi^{-3}T^{\mu\nu}\ , \quad (53)$$

$\bar{u}^{\mu}=\Phi^{-\frac{1}{2}}u^{\mu}$, $\bar{\rho}=\Phi^{-2}\rho$, $\bar{p}=\Phi^{-2}p$, $\bar{T}_{\mu\nu}=\Phi^{-1}T_{\mu\nu}$, $\bar{T}=\Phi^{-2}T$
The metric $\bar{g}_{\mu\nu}$ takes the standard FRW form

$$d\bar{s}^2=-d\bar{t}^2+\bar{a}^2(\bar{t})\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right], \quad (54)$$

where $d\bar{t}=\Phi(t)^{\frac{1}{2}}dt$ and a new scale factor $\bar{a}(\bar{t})=\Phi(\bar{t})^{\frac{1}{2}}a(\bar{t})$.

Polynomial example - Einstein frame I

In the case of the barotropic matter, the cosmological equations are

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_{\rm m}, \quad 6\frac{\ddot{\bar{a}}}{\bar{a}} = 2\bar{\rho}_\Phi - \bar{\rho}_{\rm m}(1+3w) \tag{55}$$

where

$$\bar{\rho}_\Phi = \frac{1}{2}\bar{U}(\Phi), \quad \bar{\rho}_{\rm m} = \rho_0 \bar{a}^{-3(1+w)}\Phi^{\frac{1}{2}(3w-1)} \tag{56}$$

and $w = \bar{p}_{\rm m}/\bar{\rho}_{\rm m} = p_{\rm m}/\rho_{\rm m}$. In this case, the conservation equations has the following form

$$\dot{\bar{\rho}}_{\rm m} + 3\bar{H}\bar{\rho}_{\rm m}(1+w) = -\dot{\bar{\rho}}_\Phi. \tag{57}$$

Polynomial example - Einstein frame I

and the scalar field Φ has the following form

$$\Phi(\hat{R}) = \frac{df(\hat{R})}{d\hat{R}} = \sum_{i=1}^{n} i\gamma_i \hat{R}^{i-1}. \tag{58}$$

$$\bar{U}(\hat{R}) = 2\bar{\rho}_\Phi(\hat{R}) = \frac{\sum_{i=1}^{n}(i-1)\gamma_i \hat{R}^i}{\left(\sum_{i=1}^{n} i\gamma_i \hat{R}^{i-1}\right)^2}. \tag{59}$$

Singularities $f(R) = R + \gamma R^2 + \delta R^3$ model-Jordan frame I

In the model, one finds two types of singularities, which are a consequence of the Palatini formalism: the freeze and sudden singularity. The freeze singularity appears when the multiplicative expression $\frac{b}{b+d/2}$, in the Friedmann equation (50), is equal the infinity. So we get a condition for the freeze singularity:
$2b + d = 0$ which produces a pole in the potential function. It appears that the sudden singularity in our model appears when the multiplicative expression $\frac{b}{b+d/2}$ vanishes. This condition is equivalent to the case $b = 0$.
The freeze type III singularity in our model is a solution of the algebraic equation

$$3\Omega_\gamma\Omega_\delta\Omega_R^3 + 9\Omega_\delta\Omega_R^2 + (\Omega_\gamma - 36\Omega_\delta\Omega_\Lambda)\Omega_R - 12\Omega_\gamma\Omega_\Lambda - 1 = 0 \quad (60)$$

Singularities $f(R) = R + \gamma R^2 + \delta R^3$ model-Jordan frame I

which the following solution

$$\Omega_{R_{\text{sing}}} = \Omega_\gamma^{-1}\Bigg[-1 + \frac{r(\Omega_\gamma, \Omega_\delta, \Omega_\Lambda)}{9 2^{1/3}\Omega_\delta} - \frac{2^{1/3}\left(-81\Omega_\delta^2 + 9\Omega_\gamma\Omega_\delta(\Omega_\gamma - 36\Omega_\delta\Omega_\Lambda)\right)}{9r(\Omega_\gamma, \Omega_\delta, \Omega_\Lambda)\Omega_\delta}\Bigg], \quad (61)$$

where

$$r(\Omega_\gamma, \Omega_\delta, \Omega_\Lambda) = 2\Bigg[243\Omega_\gamma^2\Omega_\delta^2(1 + 6\Omega_\gamma\Omega_\Lambda) - 729\Omega_\delta^3(1 + 6\Omega_\gamma\Omega_\Lambda) + \Big(59049\left(\Omega_\gamma^2 - 3\Omega_\delta\right)^2\Omega_\delta^4(1 + 6\Omega_\gamma\Omega_\Lambda)^2 - \left(81\Omega_\delta^2 - 9\Omega_\gamma\Omega_\delta(\Omega_\gamma - 36\Omega_\delta\Omega_\Lambda)\right)^3\Big)^{1/2}\Bigg]^{1/3}. \quad (62)$$

Singularities $f(R) = R + \gamma R^2 + \delta R^3$ model-Jordan frame I

For the sudden singularity the condition $b = 0$ provides the equation

$$1 + \Omega_{\mathrm{R}} \left[2\Omega_\gamma + 3\Omega_\delta \Omega_{\mathrm{R}}\right] = 0. \tag{63}$$

which has the following solutions

$$\Omega_{\mathrm{R_{sing}}} = \frac{-\Omega_\gamma \pm \sqrt{\Omega_\gamma^2 - 3\Omega_\delta}}{3\Omega_\delta}. \tag{64}$$

Singularities in Starobinsky model in Palatini formalism I

$$2b + d = 0 \Longrightarrow f(K, \Omega_{\Lambda,0}, \Omega_\gamma) = 0 \tag{65}$$

or

$$-3K - \frac{K}{3\Omega_\gamma(\Omega_{\mathrm{m}} + \Omega_{\Lambda,0})\Omega_{\Lambda,0}} + 1 = 0, \tag{66}$$

where $K \in [0,\ 3)$.
The solution of the above equation is

$$K_{\mathrm{freeze}} = \frac{1}{3 + \frac{1}{3\Omega_\gamma(\Omega_{\mathrm{m}}+\Omega_{\Lambda,0})\Omega_{\Lambda,0}}}. \tag{67}$$

From equation (67), we can find an expression for a value of the scale factor for the freeze singularity

$$a_{\mathrm{freeze}} = \left(\frac{1-\Omega_{\Lambda,0}}{8\Omega_{\Lambda,0} + \frac{1}{\Omega_\gamma(\Omega_{\mathrm{m}}+\Omega_{\Lambda,0})}}\right)^{\frac{1}{3}}. \tag{68}$$

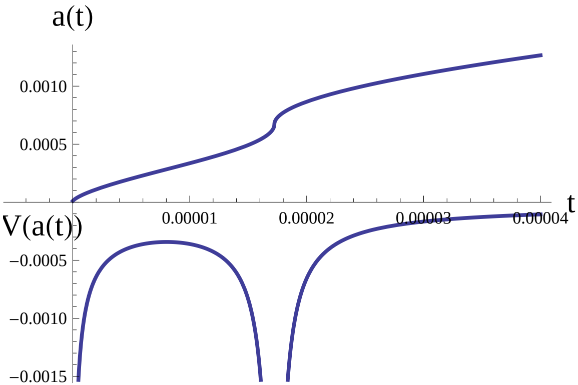


Figure: Illustration of sewn freeze singularity, when the potential $V(a)$ has a pole. Diagram of $a(t)$ is constructed from the dynamics in two disjoint region $\{a\colon a < a_s\}$ and $\{a\colon a > a_s\}$.

The sudden (type II) singularity appears when $b = 0$. This provides the following algebraic equation

$$1 + 2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + \Omega_{\Lambda,0})(K+1) = 0. \tag{69}$$

The above equation can be rewritten as

$$1 + 2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + 4\Omega_{\Lambda,0}) = 0. \tag{70}$$

From equation (70), we have the formula for the scale factor for the sudden singularity

$$a_{\mathrm{sudden}} = \left(-\frac{2\Omega_{\mathrm{m},0}}{\frac{1}{\Omega_\gamma} + 8\Omega_{\Lambda,0}}\right)^{1/3}. \tag{71}$$

which, in fact, becomes a (degenerate) critical point and a bounce at the same time.

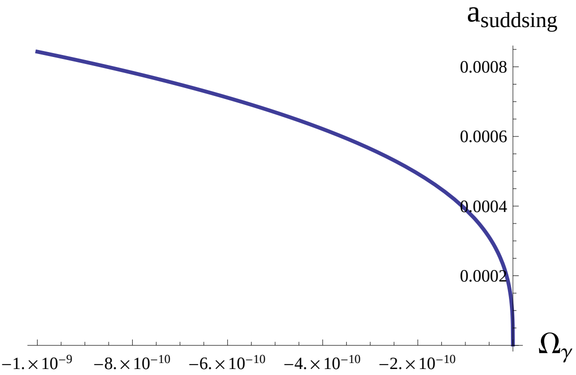


Figure: Diagram of the relation between a_{sing} and negative Ω_γ. Note that in the limit $\Omega_\gamma \mapsto 0$ the singularity overlaps with a big-bang singularity.

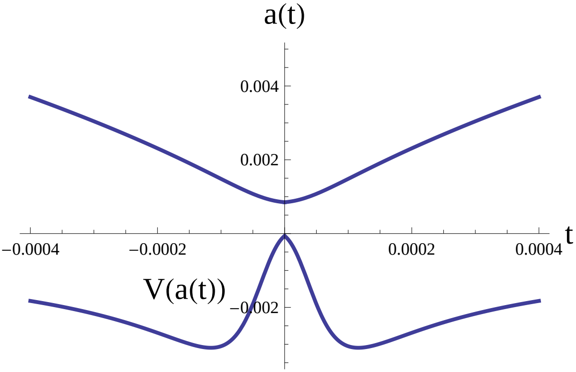
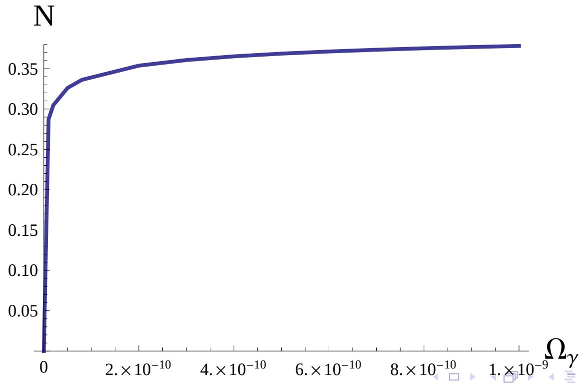


Figure: Illustration of a sewn sudden singularity. The model with negative Ω_γ has a mirror symmetry with respect to the cosmological time. Note that the spike on the diagram shows discontinuity of the function $\frac{\partial V}{\partial a}$. Note the existence of a bounce at $t = 0$.

It is also interesting that trajectories in neighbourhood of straight vertical line of freeze singularities undergo short time inflation $x = \text{const}$. The characteristic number of e-foldings from t_{init} to t_{fin} of this inflation period

$$N = H_{\text{init}}(t_{\text{fin}} - t_{\text{init}})$$

(see formula (3.13) in De Felice (2010) with respect to Ω_γ The number of e-foldings is too small for to obtain the inflation effect.



Statistical analysis of the model I

The following astronomical observations were used:

- supernovae of type Ia (Union 2.1 dataset),
- BAO (SDSS DR7 dataset, 6dF Galaxy Redshift Survey, WiggleZ measurements),
- measurements of $H(z)$ for galaxies,
- Alcock-Paczyński test,
- measurements of CMB and lensing by Planck and low ℓ polarisation by WMAP.

The total likelihood function is expressed in the following form

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)} L_{\text{CMB+lensing}}. \tag{72}$$

In estimation of model parameters, we use our own code CosmoDarkBox (the Metropolis-Hastings algorithm).

Statistical analysis of the model II

We use Bayesian information criterion (BIC), for comparison our model with the ΛCDM model. The expression for BIC is defined as

$$\mathrm{BIC} = \chi^2 + j \ln n, \tag{73}$$

where χ^2 is the value of χ^2 in the best fit, j is the number of model parameters (our model has three parameters, the ΛCDM model has two parameters) and n is number of data points ($n = 625$).

- the Starobinsky-Palatini model — $\mathrm{BIC}_{SP} = 135.668$
- the ΛCDM model $\mathrm{BIC}_{\Lambda\mathrm{CDM}} = 129.261$.

$$\Delta\mathrm{BIC} = \mathrm{BIC}_{SP} - \mathrm{BIC}_{\Lambda\mathrm{CDM}} = 6.407.$$

The evidence for the model is strong as ΔBIC is more than 6. So, in comparison to our model, the evidence in favor of the ΛCDM model is strong, but we cannot absolutely reject our model.

Table: The best fit and errors for the estimated model for the positive Ω_γ with $\Omega_{\text{m},0}$ from the interval $(0.27, 0.33)$, Ω_γ from the interval $(0.0, 2.6 \times 10^{-9})$ and H_0 from the interval (66.0 (km/(s Mpc)), 70.0 (km/(s Mpc))). $\Omega_{\text{b},0}$ is assumed as 0.048468. H_0, in the table, is expressed in km/(s Mpc). The value of reduced χ^2 of the best fit of our model is equal 0.187066 (for the ΛCDM model 0.186814).

parameter	best fit	68% CL	95% CL
H_0	68.10	$+1.07$ -1.24	$+1.55$ -1.82
$\Omega_{\text{m},0}$	0.3011	$+0.0145$ -0.0138	$+0.0217$ -0.0201
Ω_γ	9.70×10^{-11}	$+1.3480 \times 10^{-9}$ -9.70×10^{-11}	$+2.2143 \times 10^{-9}$ -9.70×10^{-11}

Conclusions I

- Palatini gravity, particularly Strobinsky type, may provide simple and viable gravity models which in solar system tests do not differ much from GR.
- Palatini cosmology, particularly Strobinsky type, may provide viable cosmological models which are comparable with the LCDM model and are able to solve some inlationary or DE, DM puzzles.
- If Ω_γ is small, then $a_{\mathrm{sudden}} = \left(-\frac{2\Omega_{\mathrm{m},0}}{\frac{1}{\Omega_\gamma}+8\Omega_{\Lambda,0}}\right)^{1/3}$ for negative Ω_γ and $a_{\mathrm{freeze}} = \left(\frac{1-\Omega_{\Lambda,0}}{8\Omega_{\Lambda,0}+\frac{1}{\Omega_\gamma(\Omega_{\mathrm{m}}+\Omega_{\Lambda,0})}}\right)^{\frac{1}{3}}$ for positive Ω_γ. These values defines the natural scale at which singularities appear in the model

Conclusions II

- In both cases of negative and positive γ one deals with a finite scale factor singularity. For negative γ it is a sudden singularity - type II. The evolutionary scenarios reveal the presence of bounce during the cosmic evolution.
- For $\gamma > 0$ it is type III freeze singularity providing pre-inflationary era.
- Two phases of deceleration and two phases of acceleration are key ingredients of our model. While the first phase models transition from the matter domination epoch to the pre-inflation the second phase models transition from the second matter dominated epoch toward the present day acceleration.

Conclusions III

- The phase portrait for model with the positive value of γ is equivalent to the phase portrait of ΛCDM model (following the dynamical system theory equivalence assumes the form of topological equivalence establish by homeomorphism). There is only a quantitative difference related with the presence of the non-isolated freeze singularity. The scale of appearance of this type singularity can be also estimated and in terms of redshift: $z_{\text{freeze}} = \Omega_{\gamma}^{-1/3}$.
- For the StarobinskyPalatini model in the Einstein frame for the positive parameter, the sewn freeze singularity are replaced by the generalized sudden singularity. In consequence this model is not equivalent to the phase portrait of the LCDM model. This model can provide a proper number of e-folds $N = 50 - 60$.

Conclusions IV

- There are also some other advantages when transforming to Einstein frame, namely that in this frame one naturally obtains the formula on dynamical dark energy which is going at late time toward cosmological constant. It is important that corresponding parametrization of dark energy is not postulated ad hock but it emerges from the first principles – which is the formulation of the problem in the Einstein frame. It is important that the parametrization of dark energy (energy density as well as a pressure) in terms of the Ricci scalar is given in a covariant form from the structure equation.

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Thank you !