Algebraic structures in exceptional geometry

Martin Cederwall

Based on:
D. Berman, MC, A. Kleinschmidt, D. Thompson, JHEP 1301 (2013) 64 [arXiv:1208.5884];
MC, J. Edlund, A. Karlsson, JHEP 1307 (2013) 028 [arXiv:1302.6736];
MC, JHEP 1307 (2013) 025 [arXiv:1302.6737];
D.S. Berman, MC, M.J. Perry, JHEP 1409 (2014) 066 [arXiv:1401.1311];
MC, JHEP 1409 (2014) 070 [arXiv:1402.2513];
MC, Fortsch. Phys. 62 (2014) 942 [arXiv:1409.4463];
MC, J. Palmkvist, JHEP 1508 (2015) 036 [arXiv:1503.06215];
MC, J.A. Rosabal, JHEP 1507 (2015) 007 [arXiv:1504.04843];
MC, JHEP 1606 (2016) 006 [arXiv:1603.04684];
G. Bossard, MC, A. Kleinschmidt, J. Palmkvist, H. Samtleben, arXiv:1708.08936;
L. Carbone, MC, J. Palmkvist, to appear;
D. Berman, MC, C. Strickland-Constable, work in progress;
D. Berman, MC, E. Malek, work in progress;
MC, J. Palmkvist, work in progress;

and work by others (Hull, Hohm, Palmkvist, Samtleben, Zwiebach,...)

9^{th} $\mathbf{M} \cap \Phi$ meeting
Belgrade, Sept 22, 2017

Duality symmetries in string theory/M-theory mix gravitational and non-gravitational fields. Manifestation of such symmetries calls for a generalisation of the concept of geometry.

It has been proposed that the compactifying space (torus) is enlarged to accommodate momenta (representing momenta and brane windings) in modules of a duality group.

This leads to *double geometry*
in the context of T-duality [Hull et al.; Hitchin;...]

and *exceptional geometry*
in the context of U-duality, [Hull; Berman et al.; Coimbra et al.;...]

The duality group is “present” already in the uncompactified theory. It becomes “geometrised”.

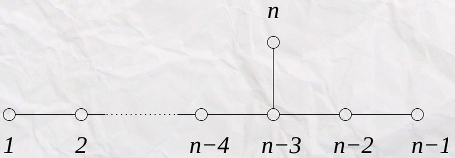
In the present talk, I will

- Describe the basics of extended geometry, with focus on the gauge transformations;
- Describe the appearance of Borcherds superalgebras and Cartan-type superalgebras (tensor hierarchy superalgebras);
- Indicate why L_∞ algebras provide a good framework for describing the gauge symmetries.

I.e., more focus on algebraic aspects, and less on geometric...

Compactify from 11 to $11 - n$ dimensions on T^n. As is well known, all fields and charges fall into modules of $E_{n(n)}$.

n	$E_{n(n)}$	
3	$SL(3) \times SL(2)$	
4	$SL(5)$	
5	$Spin(5,5)$	
6	$E_{6(6)}$	
7	$E_{7(7)}$	
8	$E_{8(8)}$	
9	$E_{9(9)}$	



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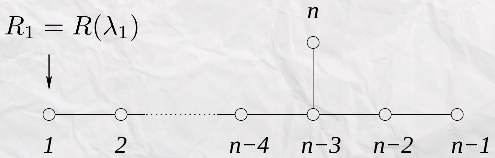
I will focus mainly on internal diffeomorphisms, and how they generalise. The ordinary diffeomorphisms go together with gauge transformations for the 3-form and (dual) 6-form fields (and for high enough n also gauge transformations for dual gravity, etc.) in an $E_{n(n)}$ module R_1. This is the "coordinate module". The derivative transforms in $\overline{R}_1$.

Compactify from 11 to $11 - n$ dimensions on T^n. As is well known, all fields and charges fall into modules of $E_{n(n)}$.

n	$E_{n(n)}$	R_1
3	$SL(3) \times SL(2)$	$(\mathbf{3}, \mathbf{2})$
4	$SL(5)$	$\mathbf{10}$
5	$Spin(5,5)$	$\mathbf{16}$
6	$E_{6(6)}$	$\mathbf{27}$
7	$E_{7(7)}$	$\mathbf{56}$
8	$E_{8(8)}$	$\mathbf{248}$
9	$E_{9(9)}$	fund



Example: E_7

Gauge parameters ξ^M in **56** of E_7:

$$\begin{array}{ccccccccc} \xi^m & & \lambda_{mn} & & \tilde{\lambda}_{mnpqr} & & \tilde{\xi}_{m,n_1\ldots n_7} & \leftarrow & \xi^M \\ 7 & + & 21 & + & 21 & + & 7 & = & 56 \end{array}$$

Fields in $E_{7(7)}/K(E_{7(7)}) = E_{7(7)}/(SU(8)/\mathbb{Z}_2)$. Dimension of coset: $133 - 63 = 70$. Parametrised by

$$\begin{array}{ccccccc} g_{mn} & & C_{mnp} & & \tilde{C}_{mnpqrs} & & \leftarrow G_{MN} \\ 28 & + & 35 & + & 7 & = & 70 \end{array}$$

From the point of view of $N = 8$ supergravity in $D = 4$, this is the scalar field coset. Now it becomes a generalised metric.

There are also mixed fields (generalised graviphotons): 1-forms in R_1, etc.

The situation for T-duality is simpler.

Compactification from 10 to $10 - d$ dimensions give the (continuous) T-duality group $O(d,d)$. The momenta are complemented with string windings to form the $2d$-dimensional module (*cf.* talks by Lj. Davidović and by Đ. Minić).

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Note that the duality group is not to be seen as a global symmetry.

Discrete duality transformations in $O(d,d;\mathbb{Z})$ or $E_{n(n)}(\mathbb{Z})$ should arise as symmetries in certain backgrounds, just as the mapping class group $SL(n;\mathbb{Z})$ arises as discrete isometries of a torus.

The rôle of the continuous versions of the duality groups should be analogous to that of $GL(n)$ in ordinary geometry (gravity).

One has to decide how tensors transform.

The generic recipe is to mimic the Lie derivative for ordinary diffeomorphisms:

$$L_U V^m = \underbrace{U^n \partial_n V^m}_{\text{transport term}} - \underbrace{\partial_n U^m V^n}_{\mathfrak{gl}\text{ transformation}}$$

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In the case of U-duality, the role of GL is assumed by $E_{n(n)} \times \mathbb{R}^+$, and

$$\mathscr{L}_U V^M = \underbrace{U^N \partial_N V^M}_{\text{transport term}} + \underbrace{Z^{MN}{}_{PQ} \partial_N U^P V^Q}_{\mathfrak{e}_{n(n)} \oplus \mathbb{R}\text{ transformation}}$$

where $Z^{MN}{}_{PQ} = -\alpha_n P^M_{\text{adj}\,Q,}{}^N{}_P + \beta_n \delta^M_Q \delta^N_P$.

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$$\begin{aligned} \mathscr{L}_U V^M &= L_U V^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q \\ &= \underset{\substack{\uparrow \\ \text{transport term}}}{U^N \partial_N V^M} + \underset{\substack{\uparrow \\ \mathfrak{e}_{n(n)} \oplus \mathbb{R}\ \text{transformation}}}{Z^{MN}{}_{PQ} \partial_N U^P V^Q} \end{aligned}$$

where $Z^{MN}{}_{PQ} = -\alpha_n P^M_{\mathrm{adj}\,Q,}{}^N{}_P + \beta_n \delta^M_Q \delta^N_P = Y^{MN}{}_{PQ} - \delta^M_P \delta^N_Q$.

The transformations form an "algebra" for $n \leq 7$:

$$[\mathscr{L}_U, \mathscr{L}_V]W^M = \mathscr{L}_{[U,V]}W^M$$

where the "Courant bracket" is $[U,V]^M = \frac{1}{2}(\mathscr{L}_U V^M - \mathscr{L}_V U^M)$, provided that the derivatives fulfil a "*section condition*".

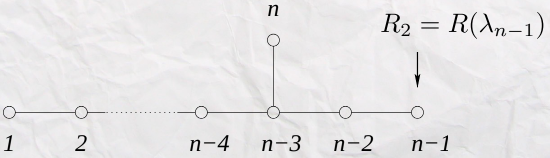
The *section condition* ensures that fields locally depend only on an n-dimensional subspace of the coordinates, on which a $GL(n)$ subgroup acts. It reads $Y^{MN}{}_{PQ}\partial_M \ldots \partial_N = 0$, or

$$(\partial \otimes \partial)|_{\overline{R}_2} = 0$$

For $n \geq 8$ more local transformations emerge.

$$(\partial \otimes \partial)|_{\overline{R_2}} = 0$$

n	R_1	R_2
3	$(\mathbf{3}, \mathbf{2})$	$(\overline{\mathbf{3}}, \mathbf{1})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$
5	$\mathbf{16}$	$\mathbf{10}$
6	$\mathbf{27}$	$\overline{\mathbf{27}}$
7	$\mathbf{56}$	$\mathbf{133}$
8	$\mathbf{248}$	$\mathbf{1} \oplus \mathbf{3875}$



The interpretation of the section condition is that the momenta locally are chosen so that they may span a linear subspace of cotangent space with maximal dimension, such that any pair of covectors p, p' in the subspace fulfil $(p \otimes p')|_{\overline{R}_2} = 0$.

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The corresponding statement for double geometry is $\eta^{MN}\partial_M \otimes \partial_N = 0$, where η is the $O(d,d)$-invariant metric. The maximal linear subspace is a d-dimensional isotropic subspace, and it is determined by a pure spinor Λ. Once a Λ is chosen, the section condition can be written $\Gamma^M \Lambda \partial_M = 0$.

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Locally, supergravity is recovered. Globally, non-geometric solutions are also obtained.

There is a universal form of the generalised diffeomorphisms for any Kac–Moody algebra and choice of coordinate representation.

Let the coordinate representation be $R(\lambda)$ (for λ a fundamental weight). Then

$$\sigma Y = -\eta_{AB} T^A \otimes T^B + (\lambda, \lambda) + \sigma - 1 \ ,$$

where η is the Killing metric and $\sigma a \otimes b = b \otimes a$.

This follows from the existence of a solution to the section constraint in the form of a linear space:

- Each momentum must be in the minimal orbit. Equivalently, $p \otimes p \in \overline{R(2\lambda)}$.
- Products of different momenta may contain $\overline{R(2\lambda)}$ and $\overline{R(2\lambda - \alpha)}$, where $R(2\lambda - \alpha)$ is the highest representation in the antisymmetric product.

Expressing these conditions in terms of the quadratic Casimir gives Y.

I will skip the detailed description of the generalised gravity. It effectively provides the local dynamics of gravity and 3-form, which are encoded by a vielbein ${E_M}^A$ in the coset $(E_{n(n)} \times \mathbb{R})/K(E_{n(n)})$.

n	$E_{n(n)}$	$K(E_{n(n)})$
3	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$
4	$SL(5)$	$SO(5)$
5	$Spin(5,5)$	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$
6	$E_{6(6)}$	$USp(8)/\mathbb{Z}_2$
7	$E_{7(7)}$	$SU(8)/\mathbb{Z}_2$
8	$E_{8(8)}$	$Spin(16)/\mathbb{Z}_2$
9	$E_{9(9)}$	$K(E_{9(9)})$

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For $n \geq 8$, the coset $E_{n(n)}/K(E_{n(n)})$ contains higher mixed tensors that do not carry independent physical degrees of freedom. They are removed by "extra" local transformations that arise in the commutator between gen. diffeomorphisms.

[Hohm, Samtleben 2014; MC, Rosabal 2015]

[Bossard, MC, Kleinschmidt, Palmkvist, Samtleben 2017]

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One may introduce (local) supersymmetry. In the case of T-duality, the superspace is based on the fundamental representation of an orthosymplectic supergroup $OSp(d,d|2s)$. The exceptional cases are unexplored, but will be based on ∞-dimensional superalgebras. [MC 2016]

Reducibility and Borcherds superalgebras

The generalised diffeomorphisms do not satisfy a Jacobi identity. On general grounds, it can be shown that the "Jacobiator"

$$[[U,V],W]+\text{cycl}\neq 0\ ,$$

but is proportional to $([U,V],W)+\text{cycl}$, where $(U,V)=\frac{1}{2}(\mathscr{L}_UV+\mathscr{L}_VU)$.

It is important to show that the Jacobiator in some sense is trivial. It turns out that $\mathscr{L}_{(U,V)}W=0$ (for $n\leq 7$), and the interpretation is that it is a gauge transformation with a parameter representing reducibility (for $n\leq 6$).

In double geometry, this reducibility is just the scalar reducibility of a gauge transformation: $\delta B_2=d\lambda_1$, with the reducibility $\delta\lambda_1=d\lambda'_0$.

In exceptional geometry, the reducibility turns out to be more complicated, leading to an infinite (but well defined) reducibility, containing the modules of tensor hierarchies, and providing a natural generalisation of forms (having connection-free covariant derivatives).

The reducibility continues, and there are ghosts at all levels > 0. The representations are those of a "tensor hierarchy", the sequence of representations R_n of n-form gauge fields in the dimensionally reduced theory.

$$R_1 \xleftarrow{\partial} R_2 \xleftarrow{\partial} R_3 \xleftarrow{\partial} \ldots$$

Example, $n = 5$:

$$\mathbf{16} \xleftarrow{\partial} \mathbf{10} \xleftarrow{\partial} \overline{\mathbf{16}} \xleftarrow{\partial} \mathbf{45} \xleftarrow{\partial} \overline{\mathbf{144}} \xleftarrow{\partial} \ldots$$

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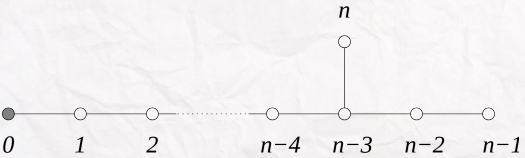
$$16 - 10 + 16 - 45 + 144 - \ldots = 11 \ ,$$

(suitably regularised) which is the number of degrees of freedom of a pure spinor.

The representations $\{R_n\}_{n=1}^{\infty}$ agree with

- The ghosts for a "pure spinor" constraint (a constraint implying an object lies in the minimal orbit);
- The positive levels of a Borcherds superalgebra $\mathscr{B}(E_n)$.



Indeed, the denominator appearing in the denominator formula for $\mathscr{B}(E_n)$ is identical to the partition function of a "pure spinor".

[MC, Palmkvist 2015]

$\mathscr{B}(D_n) \approx \mathfrak{osp}(n, n|2)$

$\mathscr{B}(A_n) \approx \mathfrak{sl}(n+1|1)$

$$\dots \xleftarrow{\partial} R_{-1} \xleftarrow{\partial} R_0 \xleftarrow{\partial} \underbrace{R_1 \xleftarrow{\partial} R_2 \xleftarrow{\partial} \dots \xleftarrow{\partial} R_{8-n}}_{\text{covariant}} \xleftarrow{\partial} R_{9-n} \xleftarrow{\partial} R_{10-n} \xleftarrow{\partial} \dots$$

The modules $R_1, \dots, R_{8-n}$ behave like forms. The "exterior derivative" is connection-free (for a torsion-free connection), and there is a wedge product.

[MC, Edlund, Karlsson 2013]

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[MC, Edlund, Karlsson 2013]

"Symmetry": $R_{9-n} = \overline{R}_n$.

There is another extension to negative levels that respects this symmetry, and seems more connected to geometry: tensor hierarchy algebras.

[Palmkvist 2013]

In the classification of finite-dimensional superalgebras by Kac, there is a special class, "Cartan-type superalgebras".

The Cartan-type superalgebra $W(n)$, which I prefer to call $W(A_{n-1})$, is asymmetric between positive and negative levels, and (therefore) not defined through generators corresponding to simple roots and Serre relations.

$W(A_{n-1})$ is the superalgebra of derivations on the superalgebra of (pointwise) forms in n dimensions.

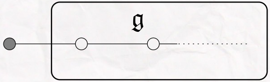
Any operation $\omega \to \Omega \wedge \imath_V \omega$, where Ω is a form and V a vector, belongs to $W(A_{n-1})$. A basis is given by

level $= 1$	$\imath_a$
0	$e^b \imath_a$
-1	$e^{b_1} e^{b_2} \imath_a$
-2	$e^{b_1} e^{b_2} e^{b_3} \imath_a$
$\ldots$	$\ldots$

A subalgebra $S(A_{n-1})$ contains traceless tensors.

The positive levels agree with $\mathscr{B}(A_{n-1}) \approx \mathfrak{sl}(n|1)$

In spite of the absence of a Cartan involution, there is a way to give a systematic Chevalley–Serre presentation of the superalgebra, based on the same Dynkin diagram as the Borcherds superalgebra.



[Carbone, MC, Palmkvist 2017 (in prep.)]

Note that the representations of torsion and torsion Bianchi identity appear at levels -1 and -2.

The construction can be extended to $W(D_n)$, and, most interestingly, $W(E_n)$ (and the corresponding $S(\mathfrak{g})$).

The statements about torsion and Bianchi identities remain true (but we still lack a geometric argument).

Back to the Jacobi identity. Expressed in terms of a fermionic ghost in R_1,

$$[[c,c],c] \neq 0$$

How is this remedied? The most general formalism for gauge symmetries is the Batalin–Vilkovisky formalism, where everything is encoded in the master equation $(S,S)=0$.

If transformations are field-independent, one may consider the ghost action consistently. An L_∞ algebra is a (super)algebraic structure which provides a perturbative solution to the master equation.

If C denotes *all* ghosts, then the master equation states the nilpotency of a transformation

$$\delta C = (S,C) = \partial C + [C,C] + [C,C,C] + [C,C,C,C] + \dots$$

The identities that need to be fulfilled are:

$$
\begin{aligned}
&\partial^2 C = 0 \ , \\
&\partial[C,C] + 2[\partial C, C] = 0 \ , \\
&\partial[C,C,C] + 2[[C,C],C] + 3[\partial C, C, C] \ , \\
&\ldots
\end{aligned}
$$

Assuming $\partial c = 0$, the non-vanishing of $[[c,c],c]$ can be compensated by the derivative of an element in R_2 (representing reducibility). One needs to introduce a 3-bracket

$$[c,c,c] \in R_2 \ .$$

Then, there are more identities to check.

For double field theory, a 3-bracket is enough. [Hohm, Zwiebach 2017]

For exceptional field theory, there are signs, that one will in fact obtain arbitrarily high brackets. There are also other issues concerning the non-covariance outside the “form window”. I will not go into detail. [Berman, MC, Strickland-C, in progr.]

The area has rich connections to various areas of pure mathematics, some of which are under investigation:

- Group theory and representation theory
- Minimal orbits
- Superalgebras
- Cartan-type superalgebras
- Infinite-dimensional (affine, hyperbolic,...) Lie algebras
- Geometry and generalised geometry
- Automorphic forms
- L_∞ algebras
- ...

- Can the formalism be continued to $n > 9$? The transformations work for E_9, and there seems to be no reason (other than mathematical difficulties) that it stops there. Is there a connection to the "E_{10} proposal" with emergent space?
- *Geometry from algebra?* What is the precise geometric relation between the tensor hierarchy algebra and the generalised diffeomorphisms?
- *Superspace/supergeometry?* And some formalism generalising that of pure spinor superfields, that manifests supersymmetry?
- *The section condition:* Can it be lifted, or dynamically generated?
- What can be learned about the full string theory / M-theory?
- …

Thank you!