Power law method for finding soliton solutions of the 2+1 Ricci flow model

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Outline

Introduction

- The paper reviews few general methods which are usually used for tackling integrable models and for finding their analytic solutions.
- The *symmetry method* and the *auxiliary equation method* will be considered. Both of them have a similar philosophy: replacing the model by an ODE obtained through similarity reduction (in the approach based on symmetry), respectively by passing to the wave variable.
- The focus will be put on the auxiliary equation method and its use in the direct finding of soliton type solutions. A general approach, unifying methods as *tanh* or $G\prime/G$, will be prposed. It will be denominated as the *power law method*.
- The proposed algorithm will be illustrated on the KdV Equation and on the Ricci flow model in 2+1 dimensions, a fruitful model in studying black holes and in the attempt of obtaining a quantum theory of gravity.

Integrability of the nonlinear differential equations

- If solutions exist, the nonlinear differential equations or the system of equations are said to be integrable.
- There is not a general theory/procedure allowing to completely solve nonlinear ODEs or PDEs. Sometimes it is quite enough to decide if the system is integrable or not.
- Main methods for deciding on integrability: Hirota' s bilinear method, Backlund transformation, Inverse scattering, Lax pair operator, Painleve analysis, Symmetry approach, Expansion method, etc. In this presentation we will focus on the last two: the symmetry approach and the expansion method applied to nonlinear PDEs.
- The *symmetry method* allows to find solutions of a "complicated" PDE, by: (i) reducing its form or the number of the degrees of freedom (till an ODE); (ii) looking for the solutions of the "reduced" equation and pull-them back into the solution of the initial PDE.
- The *expansion method* for PDEs has many versions: tanh, cosh, $(G\prime/G)$-expansion, etc. It supposes: (i) reducing PDEs ?? ODEs by passage to the *wave variable*; (ii) looking for solutions of an master equation in terms of solutions of an auxiliary equation.

Symmetries and their applications in nonlinear dynamics

- Lie symmetry method - efficient techniques in studying the integrability. It allows to obtain: (i) First integrals/invariants specific for the symmetry transformations. (ii) Classes of exact solutions through similarity reduction (reduction of PDEs to ODEs). (iii) New solutions starting from known ones.

- The *classical approach (CSM).* [Olver] for solving partial differential equations asks for the invariance of the equations to the action of an infinitesimal symmetry operator. Let us refer to an general m-th order $(1+1)$-dimensional evolution equation of the form:

$$u_t = E(t, x, u, u_x, ...u_{mx}) \ , \ \text{with} \ \ u_{kx} = \frac{\partial^k u}{\partial x^k} \ , \ 1 \leq k \leq m \tag{1}$$

$$X = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \tag{2}$$

- The method supposes to find the symmetries $\xi^i(x, u)$ and $\phi_\alpha(x, u)$ which leave invariant the class of solutions for (1)

The auxiliary equation method

The idea: replacing a PDE with an ODE. Steps followed:

- introduction of the *wave variable* $\xi - \xi(t, x^1, ..., x^p)$
- looking for solutions of the ODE we got in terms of the solutions of another ODE, called *auxiliary equation*, with already known solutions.

Let us consider:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, ...) = 0 \tag{3}$$

We define the wave coordinate:

$$\xi = x - Vt \tag{4}$$

By that, the equation (3) becomes the following ODE:

$$Q(u, u', u'', u''', ...) = 0 \tag{5}$$

where the derivatives are considered in respect with ξ. There are many versions related to the auxiliary equation:

- The tanh *method* - solutions of (5) in terms of $\tanh\xi$, $\cosh\xi, \sinh\xi$, etc. which are solutions $\varphi(\xi)$ of equations, as Riccati, so:

$$u(\xi) =^{N}_{i=0} a_i \varphi^i \tag{6}$$

- the G'/G method, where $G(\xi)$ solution of an auxiliary equation. In this case:

$$u(\xi) =^{N}_{i=0} a_i \left(\frac{G'}{G}\right)^i \tag{7}$$

The polynomial expansion method / General approach

The approach we are proposing is an unifying one. More precisely, the solution of the master equation will be asked to be a polynomial expansion in terms of the solutions $G(\xi)$ of the auxiliary equation:

$$u(\xi) = \sum_{i=0}^{N} P_i(G)(G')^i \tag{8}$$

where $P_i(G)$ are polynomials in G to be determined.

The polynomial expansion method / The auxiliary equation

- Computing the derivatives of $u(\xi)$ higher order derivatives G', G'', G''', ... could appear. So we might look to a more general solution depending on higher derivatives of $G(\xi)$:

$$u(\xi) = P_0(G) + P_1(G)G' + P_2(G, G')G'' + ... \tag{9}$$

  Although, the higher derivatives $G''.G''', ...$ can be expressed in terms of G, G' by using an adequate auxiliary equation. Its choice (its order) is very important.

- **Examples of auxiliary equations**: - Riccati Equation (first order nonlinear equation):

$$G' = \alpha + \beta G^2 \tag{10}$$

  - Second order linear ODE:

$$G'' + AG' + BG = 0 \tag{11}$$

  -Second order nonlinear ODE:

$$AGG'' - B(G')^2 - CGG' - EG^2 = 0 \tag{12}$$

  -Third order nonlinear ODE:

$$AG^2G''' - B\left(G'\right)^3 - CG\left(G'\right)^2 - DG^2G' - FG^3 = 0 \tag{13}$$

The polynomial expansion method / Balancing Procedure

- Another important step:to determine the limit N of the expansion (8) by a standard "balancing" procedure: replace (8) in (5) and take into account the higheast nonlinearity and the term with the maximal order of derivation.
- In our case a new requirement is imposed: polynomial expantions for the functions $P_0(G), P_1(G), P_2(G)$,... To have a true balance and compatibility, we have to consider expansions of the form:

$$P_2(G) = \sum_{i=-2}^{0} a_i G^i \tag{14}$$

$$P_1(G) = \sum_{j=-1}^{0} b_j G^j \tag{15}$$

$$P_0(G) = c_0 \tag{16}$$

- From the algebraic system generated by these choices, we can determine the coefficients a_i,b_j and c_0. After that, we can write down the form of the solutions $u(\xi)$. These solutions have to be discussed for various possible values of the coefficients A, B, C,...appearing in the master equation.

The example of the *KdV*

- Let's consider Korteweg de Vries Eq.:

$$u_t + uu_x + \delta u_{xxx} = 0 \tag{17}$$

- We pass to the wave coordinate and after a first direct integration in respect with ξ, we get the ODE:

$$\delta u''(\xi) + \frac{1}{2}u^2(\xi) - Vu(\xi) + k = 0 \tag{18}$$

- The balancing has to be done between $\delta u''(\xi)$ and $\frac{1}{2}u^2(\xi)$.It leads to $N = 2$, that is the solution of (18) has to be considered as:

$$u(\xi) = \sum_{i=0}^{2} P_i(G)(G')^i \tag{19}$$

KdV/ Auxiliary equation

- We will consider that $G(\xi)$ is solution of the auxiliary equation

$$G'' + AG' + BG = 0 \tag{20}$$

- It is well known that the solutions of (20) depend on the values of the coefficients A, B, and three specific cases have to be considered: (i) if $\Delta = A^2 - 4B > 0$ it will be a hyperbolic solution:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(A_1 ch\frac{\sqrt{\Delta}}{2}\xi + A_2 sh\frac{\sqrt{\Delta}}{2}\xi\right) \tag{21}$$

(ii) if $\Delta = A^2 - 4B < 0$ the solution will be expressed through trigonometric functions:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(A_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + A_2 \sin\frac{\sqrt{-\Delta}}{2}\xi\right) \tag{22}$$

(iii) if $\Delta = A^2 - 4B = 0$ the solution will be:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(A_1 + A_2\xi\right) \tag{23}$$

In all the cases, A_1 and A_2 are arbitrary constants.

KdV/ Determining system for polynomials

By computing u'', using (19), (20), and equating with zero the coefficients of the various monomials in G'we get the following system of equations for P_k, $k = 0, 1, 2$:

$$2\delta P_2''(G) + P_2^2(G) = 0 \tag{24}$$

$$\delta P_1''(G) - 5\delta P_2'(G) + P_1(G)P_2(G) = 0 \tag{25}$$

$$\delta P_0''(G) - 3\delta A P_1'(G) - 5\delta BGP_2'(G) + 2\delta(2A^2 - B)P_2(G) + \frac{1}{2}P_1^2(G) + P_0(G)P_2(G) - VP_2(G) = 0 \tag{26}$$

$$-\delta AP_0'(G) - 3\delta BP_1'(G)G + \delta(A^2 - B)P_1(G) + 6\delta ABGP_2(G) + P_0(G)P_1(G) - VP_1(G) = 0 \tag{27}$$

$$-\delta BGP_0'(G) + \frac{1}{2}P_0^2(G) - VP_0(G) + \delta ABGP_1(G) + 2\delta B^2G^2P_2(G) + k = 0 \tag{28}$$

KdV/ Determining system for polynomials

It is not an algebraic system as in the (G'/G) approach, but a system of differential equations for these polynoms. The system can be solved starting from the highest order and keeping always in mind the compaticility requirement, which will ask for specific dependency of P_k.

- For the first equation (24) we can have the solution:

$$P_2(G) = \sum_{i=-2}^{0} a_i G^i \tag{29}$$

It is easy to get from (24) the values of the coefficients a_i:

$$a_0 = 0, \;\; a_1 = 0, \;\; a_2 = -12\delta \tag{30}$$

- Similarly, from (25) we get:

$$P_1(G) = \sum_{j=-1}^{0} b_j G^j \tag{31}$$

The constants appearing in (31) will be:

$$b_0 = 0, \;\; b_1 = -12A \tag{32}$$

KdV/ Determining system for polynomials

- The final results are:

$$P_0(G) = -8B \tag{33}$$

$$P_1(G) = -12A\frac{1}{G} \tag{34}$$

$$P_2(G) = -12\delta\frac{1}{G^2} \tag{35}$$

- The solution $u(\xi)$ of KdV equation (17) will be:

$$u(\xi) = -8B - 12A\frac{G'}{G} - 12\delta\left(\frac{G'}{G}\right)^2 \tag{36}$$

Ricci 2*D* equation

- The Ricci flow equation in 2*D* which has the form:

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \tag{37}$$

- With the wave transformation, the equation (37) takes the form:

$$U' U^2 + \frac{\alpha\beta}{v}(UU'' - U'^2) = 0 \tag{38}$$

- The focus will be put on the auxiliary equation method and its use in the direct finding of soliton type solutions. A general approach, unifying methods as *tanh* or $G\prime/G$, will be prposed. It will be denominated as the *power law method*.

- We will solve equation (38) by four different methods, in order to compare the solutions themselves and the efficiency of the methods.

Ricci $2D$ equation/ The direct integration

- The equation (38) can be solved directly by double integration and the form of the solution is:

$$\begin{aligned} U &= \frac{e^{\lambda}}{-1 + v c_1 e^{\lambda}} \\ \lambda &= \frac{\xi + c_2}{c_1 \alpha \beta} \end{aligned} \quad (39)$$

- The direct integration leads to singular solution which are not of Physical interest.

Ricci 2*D* equation/Solution of *tanh* type

- The simplest way of finding soliton solutions for (13) is to use Riccati as auxiliary equation and to look for solutions of (38) . More precisely, we will consider:

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i \tag{40}$$

  The Riccati equation has the form:

$$\varphi' = k + \varphi^2 \tag{41}$$

  with k a real constant.

- Considering integrating constant as zero,

$$\begin{aligned}
\xi &= \begin{cases} \frac{1}{\sqrt{k}} \tan^{-1}\left(\frac{\varphi}{\sqrt{k}}\right) \\ -\frac{1}{\sqrt{k}} \cot^{-1}\left(\frac{\varphi}{\sqrt{k}}\right) \end{cases}, \quad k > 0 \\
\xi &= -\frac{1}{\varphi}, \quad k = 0 \\
\xi &= \begin{cases} -\frac{1}{\sqrt{-k}} \tanh^{-1}\left(\frac{\varphi}{\sqrt{-k}}\right) \\ -\frac{1}{\sqrt{-k}} \coth^{-1}\left(\frac{\varphi}{\sqrt{-k}}\right) \end{cases}, \quad k < 0
\end{aligned} \tag{42}$$

Solution of *tanh* type

- The balancing procedure leads to the maximal value $N = 1$, that is the solution we are looking for will have the form:

$$U(\xi) = a_0 + a_1\varphi \tag{43}$$

$$U(\xi) = \frac{\alpha\beta}{v}(\sqrt{-k} - \sqrt{-k}\tanh\sqrt{-k}\xi) \tag{44}$$

Solution of the tanh type

Inverting the last relations will result:

$$
\begin{aligned}
\varphi &= \begin{cases} \sqrt{k}\tan\sqrt{k}\xi \\ -\sqrt{k}\cot\sqrt{k}\xi \end{cases}, \; k > 0 \\
\varphi &= -\frac{1}{\xi}, k = 0 \\
\varphi &= \begin{cases} -\sqrt{-k}\tanh\sqrt{-k}\xi \\ -\sqrt{-k}\coth\sqrt{-k}\xi \end{cases}, \; k < 0
\end{aligned} \tag{45}
$$

The balancing procedure leads to the maximal value $N = 1$, that is the solution we are looking for will have the form:

$$U(\xi) = a_0 + a_1\varphi \tag{46}$$

$$U(\xi) = \frac{\alpha\beta}{v}(\sqrt{-k} - \sqrt{-k}\tanh\sqrt{-k}\xi) \tag{47}$$

Solution of the tanh type

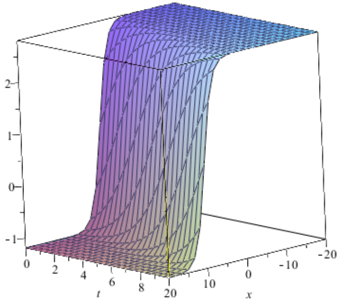

Fig.3: Ricci tanh y0

Solution of G'/G type

We solve now the equation (13) by using the G'/G method. It imposes to look for solutions of the form:

$$U(\xi) = \sum_{i=0}^{N} d_i \left(\frac{G'}{G}\right)^i \tag{48}$$

We will consider that d_i are constant coefficients, while this time, $G(\xi)$ is a solution of the auxiliary equation of the form:

$$G'' + mG' + nG = 0 \tag{49}$$

Again, the balancing procedure leads to the same limit $N = 1$.

Solution of G'/G type

By introducing (14) in (13) we get a polynomial equation in G' containing monomyals until G'^7. Equating with zero the coefficients for all this monomials we get a system of 8 ODE with the unknown quantities $a_0(G)$,$a_1(G)$,

$$a_0 = \frac{\alpha\beta m}{a_1} + Ce^{-\frac{a_1 v}{\alpha\beta}G} \tag{50}$$

$$a_1' a_1 + \frac{\alpha\beta}{v}(a_1'' a_1 - a_1') = 0 \tag{51}$$

Solution by G'/G method

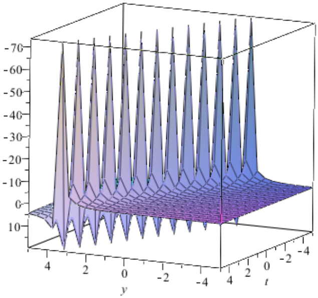

Fig. 1: Ricci Power Law x0

Ricci $2D$ model/ Polynomial expansion

- Let us now consider the same "master" equation (38), with the same "auxiliary" equation (**??**), but looking for polynomial solutions of the type (**??**). Following the general algorithm we proposed, the balancing procedure leads to the same limit $N = 1$ as in the G'/G case.
- By introducing (39) in (38) we get a polynomial equation in G' containing monomyals until G'^7. Equating with zero the coefficients for all this monomials we get a system of 8 ODE with the unknown quantities $a_0(G)$, $a_1(G)$,

$$a_0 = \frac{\alpha\beta m}{a_1} + Ce^{-\frac{a_1 v}{\alpha\beta}G} \qquad (52)$$

$$a_1' a_1 + \frac{\alpha\beta}{v}(a_1'' a_1 - a_1') = 0 \qquad (53)$$

- The final solution is quite similar with the one we got in the G'/G approach and it is presented in the figure below.

Solution by polynomial expansion

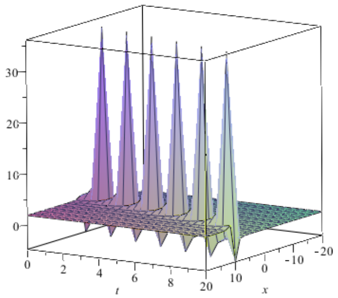

Fig. 2: Ricci Power Law y0

Concluding remarks

- We proposed a general algorithm for finding solutions of nonlinear PDEs by using polynomial expansions in terms of auxiliary equations' solutions.It includes all the methods proposed in literature, known as $tanh, \cosh, \sinh, G'/G$, etc.
- The main idea is quite similar with what symmetry method offers: to reduce a complicated equation to a simpler one, to solve this last equation, and to transfer its solutions to the master (complicated) equation.
- We pointed out the importance of three main factors: - the choice of the auxiliary equation; - the choice of the form of solution; - the balancing procedure.
- For the specific models we tackled, we get that the polynomials from (8) have the form:

  $$P_2(G) = a_2 G^{-2}$$
  $$P_1(G) = b_1 G^{-1}$$
  $$P_0(G) = c_0$$

  It appears in a natural way that really the largest class of solutions can be expressed as (G'/G) expantions. Why this expansion is choosen is not at all clear in previous approaches.
- The method is purelly analitic and it open the doors for finding other solutions which do not belong to the class of (G'/G) class.

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