Cosmological perturbations in nonlocal gravity

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Nonlocal Modified Gravity

Our action is given by

$$S = \frac{1}{16\pi G}\int\Big(R - 2\Lambda + R^p\mathcal{F}(\Box)R^q\Big)\sqrt{-g}\,\mathrm{d}^4x$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$.

We use Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\Big),\ k\in\{-1,0,1\}.$$

Equations of motion

Equation of motion are

$$
\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}R^p\mathcal{F}(\Box)R^q + R_{\mu\nu}W - K_{\mu\nu}W + \frac{1}{2}\Omega_{\mu\nu} = -(G_{\mu\nu} + \Lambda g_{\mu\nu}),\\
&\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(g_{\mu\nu}\nabla^\alpha\Box^l R^p \nabla_\alpha \Box^{n-1-l}R^q\\
&- 2\nabla_\mu\Box^l R^p\nabla_\nu\Box^{n-1-l}R^q + g_{\mu\nu}\Box^l R^p\Box^{n-l}R^q\big),\\
&K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box,\\
&W = pR^{p-1}\mathcal{F}(\Box)R^q + qR^{q-1}\mathcal{F}(\Box)R^p.
\end{aligned}
$$

Trace and 00-equations

In case of *FRW* metric there are two linearly independent equations. The most convenient choice is trace and 00 equations:

$$
\begin{aligned}
&-2R^p\mathcal{F}(\Box)R^q + RW + 3\Box W + \frac{1}{2}\Omega = R - 4\Lambda, \\
&\frac{1}{2}R^p\mathcal{F}(\Box)R^q + R_{00}W - K_{00}W + \frac{1}{2}\Omega_{00} = \Lambda - G_{00}, \\
&\Omega = g^{\mu\nu}\Omega_{\mu\nu}.
\end{aligned}
$$

Cosmological solutions with constant scalar curvature

Let $R = R_0 = const$ and we obtain

$$6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = R_0.$$

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Change of variable $b(t) = a^2(t)$ implies

$$3\ddot{b} - R_0 b = -6k.$$

Depending on the sign of the scalar curvature R_0 we obtain the following solutions for $b(t)$

$$\begin{array}{ll} R_0 > 0 & b(t) = \frac{6k}{R_0} + \sigma e^{\sqrt{\frac{R_0}{3}}t} + \tau e^{-\sqrt{\frac{R_0}{3}}t} \\ R_0 = 0 & b(t) = -k^2 t + \sigma t + \tau \\ R_0 < 0 & b(t) = \frac{6k}{R_0} + \sigma \cos\sqrt{\frac{-R_0}{3}}t + \tau \sin\sqrt{\frac{-R_0}{3}}t \end{array}$$

Cosmological solutions with constant scalar curvature

Since $R = R_0 = const$ trace and 00 equations are simplified to

$$f_0 R_0^{p+q-1}(p+q-2) = R_0 - 4\Lambda,$$

$$f_0 R_0^{p+q-1}(\frac{1}{2}R_0 + (p+q)R_{00}) = \Lambda - G_{00}.$$

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The system has a solution iff

$$R_0^{p+q-1}(R_0 + 4R_{00})(R_0 + (2\Lambda - R_0)(p+q)) = 0.$$

note that R_{00} is expressed in terms of $b(t)$ as

$$R_{00} = -\frac{3\ddot{a}}{a} = \frac{3((\dot{b})^2 - 2b\ddot{b})}{4b^2}.$$

Cosmological solutions with constant scalar curvature

In the first case, condition $R_0 + 4R_{00} = 0$ yields restrictions on values of parameters σ and τ:

$$
\begin{aligned}
R_0 > 0 &\qquad 9k^2 = R_0^2 \sigma\tau, \\
R_0 = 0 &\qquad \sigma^2 + 4k\tau = 0, \\
R_0 < 0 &\qquad 36k^2 = R_0^2(\sigma^2 + \tau^2).
\end{aligned}
$$

Case 1: $R_0 < 0$

Let $k = -1$, define φ by $\sigma = \frac{-6}{R_0}\cos\varphi$ and $\tau = \frac{-6}{R_0}\sin\varphi$, then $a(t)$ and $b(t)$ simplifies to

$$b(t) = \frac{-12}{R_0}\cos^2\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right),$$

$$a(t) = \sqrt{\frac{-12}{R_0}}\left|\cos\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right)\right|.$$

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Let $k = +1$ $b(t)$ is transformed into

$$b(t) = \frac{12}{R_0}\sin^2\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right),$$

which is nonpositive, and there is no solutions.

Case 2: $R_0 = 0$

Let $k = 0$ then functions $a(t)$ are $b(t)$ constant and we get Minkowski spacetime.

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Let $k = 0$ then functions $a(t)$ are $b(t)$ constant and we get Minkowski spacetime.
Let $k = \pm 1$, then $b(t)$ takes the form

$$b(t) = -k(t - \frac{\sigma}{2k})^2.$$

Therefore, if $k = 1$ there is no solutions, and if $k = -1$ we have $a(t) = |t + \frac{\sigma}{2}|$.

Case 3: $R_0 > 0$

If $k = 0$ we obtain a solution with constant Hubble parameter. Moreover, if $k = +1$ we choose φ such that $\sigma + \tau = \frac{6}{R_0} \cosh \varphi$ and $\sigma - \tau = \frac{6}{R_0} \sinh \varphi$. Then

$$b(t) = \frac{12}{R_0} \cosh^2 \frac{1}{2}\left(\sqrt{\frac{R_0}{3}} t + \varphi\right),$$

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In the last possibility $k = -1$, $b(t)$ takes the form

$$b(t) = \frac{12}{R_0} \sinh^2 \frac{1}{2}\left(\sqrt{\frac{R_0}{3}} t + \varphi\right),$$

$$a(t) = \sqrt{\frac{12}{R_0}} \left|\sinh \frac{1}{2}\left(\sqrt{\frac{R_0}{3}} t + \varphi\right)\right|.$$

Case 4: $R_0^{p+q-1}(R_0 + (2\Lambda - R_0)(p+q)) = 0$

If $p+q \geq 1$ then the only solution is $R_0 = 0$.
If $p+q = 0$ there is no solutions.
If $p+q \neq 0, 1$ then $R_0 = \frac{2\Lambda(p+q)}{p+q-1}$.

Perturbations

Let us consider the case $k=0$, $a(t)=e^{\lambda t}$.
We introduce the conformal time $\mathrm{d}\tau = a(t)\mathrm{d}t$, and then $a(\tau) = -\frac{1}{\lambda\tau}$.

$$ds^2 = a^2(\eta)\big(-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big)$$

Perturbations

We take the scalar perturbations of the metric in the form $\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$

$$h_{\mu\nu} = a(\eta)^2 \begin{pmatrix} -2\phi & -(\nabla B)^T \\ -\nabla B & -2\psi Id + 2\,\mathrm{Hess}\, E \end{pmatrix}$$

- ϕ, ψ, B and E depend on η, x, y, z.
- gauge transformation can make any two of those functions vanish.
- gauge invariant variables (Bardeen potentials) $\Phi = \phi - \frac{a'}{a}(B + E') - (B' + E'')$, $\Psi = \psi + \frac{a'}{a}(B + E')$,

Perturbation of the scalar curvature takes the form

$$\hat{R} = R + \delta R,$$
$$\delta R = -R_{\mu\nu}h^{\mu\nu} + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)h^{\mu\nu},$$

Perturbations

Perturbations of the equations of motion up to linear order take form

$$-m^2\delta G^\mu_\nu + (R^\mu_\nu - K^\mu_\nu)v(\Box)\delta R = 0,$$

where $m^2 = 2 + 2f_0(\mathcal{G}'\mathcal{H} + \mathcal{H}'\mathcal{G})$ i
$v(\Box) = -2(\mathcal{G}''\mathcal{H} + \mathcal{H}''\mathcal{G})f_0 + 2\mathcal{G}'\mathcal{H}'\mathcal{F}(\Box)$.

Trace of the pervious equation is

$$[m^2 + (R + 3\Box)v(\Box)]\delta R = \mathcal{U}(\Box)\delta R = 0.$$

To solve the trace equation we use Weierstrass factorization theorem

$$\mathcal{U}(\Box)\delta R = \prod_i (\Box - \omega_i^2)e^{\gamma(\Box)}\delta R = 0,$$

where ω_i^2 are the roots of the equation $\mathcal{U}(\omega^2) = 0$ and $\gamma(\Box)$ is entire function. Moreover, we assume that there is no multiple roots.

Perturbations

Roots ω_i^2 are obtained as solutions of the eigenvalue problem

$$(\Box - \omega_i^2)\delta R = 0.$$

Eigenfunctions that correspond to eigenvalue ω_i^2 are denoted δR_i.
General solution for δR is the sum over all values of ω_i^2 ie.
$\delta R = \sum_i \delta R_i$.
Eigenfunctions take the form

$$\delta R_i = (-k\tau)^{3/2} \left(C_{1i} J_{\nu_i}(-k\tau) + C_{2i} Y_{\nu_i}(-k\tau) \right),$$

where $J,\ Y$ are Bessel functions of the first and second kind resectively and $\nu_i = \sqrt{\frac{9}{4} - \frac{\omega_i^2}{H^2}}$.

Bardeen potentials

Bardeen potentials are derived from the following equations

$$-m^2(\Phi - \Psi) + v(\Box)\delta R = 0,$$
$$\delta R + (R + 3\Box)(\Phi - \Psi) = 0.$$

Then Bardeen potentials take the form

$$\Phi + \Psi = \eta(c_1(\cos(\eta) + \eta\sin(\eta)) + c_2(-\eta\cos(\eta) + \sin(\eta))) \ ,$$
$$\Phi - \Psi = \frac{1}{m^2}\sum_i v(\omega_i^2)\delta R_i,$$

where $\eta = \frac{k\tau}{\sqrt{3}}$.

Perturbations

Asymptotic behavior of the Bessel function implies that Bardeen potentials are bounded if

$$\Re\nu < \frac{3}{2}.$$

$$R - 4\Lambda + f_0 R^{p+q}(2 - p - q) = 0.$$

This polynomial equation can be explicitly solved for R if $-3 \leq p + q \leq 4$. Necessary condition for the solution to be stable is

$$1 + R^{p+q-1}(p+q)(2-p-q)f_0 < 0.$$

Note that if $p + q = 0$ or $p + q = 2$ there is no stable solutions. When $p + q = 1$ the stable solution might exist if $\Lambda < 0$ and $f_0 < 0$.

Perturbations

Pervious two conditions are reformulated

$$1 - s + u = 0, \qquad 1 + uz < 0,$$

where $s = \frac{4\Lambda}{R}, z = p + q, u = f_0 R^{z-1}(2 - z)$. This system is very simple, but does not have clear physical interpretation.

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Thank you for your attention!