Cosmological perturbations in nonlocal gravity

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Nonlocal Modified Gravity

Our action is given by

\[ S = \frac{1}{16\pi G} \int \left( R - 2\Lambda + R^p F(\Box) R^q \right) \sqrt{-g} \, d^4x \]

where \( \Box = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \), \( F(\Box) = \sum_{n=0}^{\infty} f_n \Box^n \).

We use Friedmann-Lemaître-Robertson-Walker (FLRW) metric

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad k \in \{-1, 0, 1\}. \]
Equations of motion

Equation of motion are

\[- \frac{1}{2} g_{\mu\nu} R^p \mathcal{F}(\Box) R^q + R_{\mu\nu} W - K_{\mu\nu} W + \frac{1}{2} \Omega_{\mu\nu} = -(G_{\mu\nu} + \Lambda g_{\mu\nu}),\]

\[\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( g_{\mu\nu} \nabla^{\alpha} \Box^l R^p \nabla_{\alpha} \Box^{n-1-l} R^q \right.\]
\[- 2 \nabla_\mu \Box^l R^p \nabla_\nu \Box^{n-1-l} R^q + g_{\mu\nu} \Box^l R^p \Box^{n-l} R^q \right),\]

\[K_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box,\]

\[W = pR^{p-1} \mathcal{F}(\Box) R^q + qR^{q-1} \mathcal{F}(\Box) R^p.\]
In case of FRW metric there are two linearly independent equations. The most convenient choice is trace and 00 equations:

\[-2R^p \mathcal{F}(\Box) R^q + RW + 3\Box W + \frac{1}{2} \Omega = R - 4\Lambda,\]

\[\frac{1}{2} R^p \mathcal{F}(\Box) R^q + R_{00} W - K_{00} W + \frac{1}{2} \Omega_{00} = \Lambda - G_{00},\]

\[\Omega = g^{\mu\nu} \Omega_{\mu\nu}.\]
Let $R = R_0 = \text{const}$ and we obtain

$$6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = R_0.$$
Cosmological solutions with constant scalar curvature

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Change of variable $b(t) = a^2(t)$ implies

$$3\ddot{b} - R_0 b = -6k.$$ 

Depending on the sign of the scalar curvature $R_0$ we obtain the following solutions for $b(t)$

- $R_0 > 0$ 
  $$b(t) = \frac{6k}{R_0} + \sigma e^{\sqrt{\frac{R_0}{3}}t} + \tau e^{-\sqrt{\frac{R_0}{3}}t}$$

- $R_0 = 0$ 
  $$b(t) = -k^2 t + \sigma t + \tau$$

- $R_0 < 0$ 
  $$b(t) = \frac{6k}{R_0} + \sigma \cos \sqrt{\frac{-R_0}{3}}t + \tau \sin \sqrt{\frac{-R_0}{3}}t$$
Cosmological solutions with constant scalar curvature

Since $R = R_0 = \text{const}$ trace and 00 equations are simplified to

$$f_0 R_0^{p+q-1}(p + q - 2) = R_0 - 4\Lambda,$$

$$f_0 R_0^{p+q-1}\left(\frac{1}{2}R_0 + (p + q)R_{00}\right) = \Lambda - G_{00}.$$
Cosmological solutions with constant scalar curvature

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\[
\begin{align*}
    f_0 R_0^{p+q-1}(p + q - 2) &= R_0 - 4\Lambda, \\
    f_0 R_0^{p+q-1} \left( \frac{1}{2} R_0 + (p + q)R_{00} \right) &= \Lambda - G_{00}.
\end{align*}
\]

The system has a solution iff

\[
R_0^{p+q-1}(R_0 + 4R_{00})(R_0 + (2\Lambda - R_0)(p + q)) = 0.
\]

note that $R_{00}$ is expressed in terms of $b(t)$ as

\[
R_{00} = -\frac{3\ddot{a}}{a} = \frac{3((\dot{b})^2 - 2b\dot{b})}{4b^2}.
\]
In the first case, condition $R_0 + 4R_{00} = 0$ yields restrictions on values of parameters $\sigma$ and $\tau$:

\begin{align*}
R_0 > 0 & \quad 9k^2 = R_0^2 \sigma \tau, \\
R_0 = 0 & \quad \sigma^2 + 4k \tau = 0, \\
R_0 < 0 & \quad 36k^2 = R_0^2 (\sigma^2 + \tau^2).
\end{align*}
Case 1: $R_0 < 0$

Let $k = -1$, define $\varphi$ by $\sigma = \frac{-6}{R_0} \cos \varphi$ and $\tau = \frac{-6}{R_0} \sin \varphi$, then $a(t)$ and $b(t)$ simplifies to

$$b(t) = \frac{-12}{R_0} \cos^2 \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right),$$

$$a(t) = \sqrt{\frac{-12}{R_0}} |\cos \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right)|.$$
Case 1: $R_0 < 0$

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\[
b(t) = \frac{-12}{R_0} \cos^2 \left( \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right) \right),
\]
\[
a(t) = \sqrt{-\frac{12}{R_0}} \left| \cos \left( \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right) \right) \right|.
\]

Let $k = +1$ $b(t)$ is transformed into

\[
b(t) = \frac{12}{R_0} \sin^2 \left( \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right) \right),
\]

which is nonpositive, and there is no solutions.
Case 2: $R_0 = 0$

Let $k = 0$ then functions $a(t)$ are $b(t)$ constant and we get Minkowski spacetime.
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Let $k = \pm 1$, then $b(t)$ takes the form

$$b(t) = -k(t - \frac{\sigma}{2k})^2.$$ 

Therefore, if $k = 1$ there is no solutions, and if $k = -1$ we have $a(t) = |t + \frac{\sigma}{2}|$. 
Case 3: $R_0 > 0$

If $k = 0$ we obtain a solution with constant Hubble parameter. Moreover, if $k = +1$ we choose $\varphi$ such that $\sigma + \tau = \frac{6}{R_0} \cosh \varphi$ and $\sigma - \tau = \frac{6}{R_0} \sinh \varphi$. Then

$$b(t) = \frac{12}{R_0} \cosh^2 \frac{1}{2} \left( \frac{\sqrt{R_0}}{3} t + \varphi \right),$$

$$a(t) = \sqrt{\frac{12}{R_0}} \cosh \frac{1}{2} \left( \frac{\sqrt{R_0}}{3} t + \varphi \right).$$
Case 3: $R_0 > 0$

If $k = 0$ we obtain a solution with constant Hubble parameter. Moreover, if $k = +1$ we choose $\varphi$ such that $\sigma + \tau = \frac{6}{R_0} \cosh \varphi$ and $\sigma - \tau = \frac{6}{R_0} \sinh \varphi$. Then

\[
 b(t) = \frac{12}{R_0} \cosh^2 \frac{1}{2} (\sqrt{\frac{R_0}{3}} t + \varphi),
\]

\[
 a(t) = \sqrt{\frac{12}{R_0}} \cosh \frac{1}{2} (\sqrt{\frac{R_0}{3}} t + \varphi).
\]

In the last possibility $k = -1$, $b(t)$ takes the form

\[
 b(t) = \frac{12}{R_0} \sinh^2 \frac{1}{2} (\sqrt{\frac{R_0}{3}} t + \varphi),
\]

\[
 a(t) = \sqrt{\frac{12}{R_0}} |\sinh \frac{1}{2} (\sqrt{\frac{R_0}{3}} t + \varphi)|.
\]
Case 4: $R_0^{p+q-1}(R_0 + (2\Lambda - R_0)(p + q)) = 0$

If $p + q \geq 1$ then the only solution is $R_0 = 0$.
If $p + q = 0$ there is no solutions.
If $p + q \neq 0, 1$ then $R_0 = \frac{2\Lambda(p+q)}{p+q-1}$.
Let us consider the case $k = 0$, $a(t) = e^{\lambda t}$.
We introduce the conformal time $d\tau = a(t)dt$, and then $a(\tau) = -\frac{1}{\lambda \tau}$.

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)$$
Perturbations

We take the scalar perturbations of the metric in the form
\[ \hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \]

\[ h_{\mu\nu} = a(\eta)^2 \begin{pmatrix} -2\phi & -(\nabla B)^T \\ -\nabla B & -2\psi I + 2 \text{Hess} E \end{pmatrix} \]

- \( \phi, \psi, B \) and \( E \) depend on \( \eta, x, y, z \).
- gauge transformation can make any two of those functions vanish.
- gauge invariant variables (Bardeen potentials)
  \[ \Phi = \phi - \frac{a'}{a} (B + E') - (B' + E''), \quad \Psi = \psi + \frac{a'}{a} (B + E') \]

Perturbation of the scalar curvature takes the form
\[ \hat{R} = R + \delta R, \]
\[ \delta R = -R_{\mu\nu} h^{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) h^{\mu\nu}, \]
Perturbations of the equations of motion up to linear order take form

\[-m^2 \delta G^\mu_\nu + (R^\mu_\nu - K^\mu_\nu)v(\Box)\delta R = 0,\]

where \(m^2 = 2 + 2f_0(G'\mathcal{H} + \mathcal{H}'G)\) i
\(v(\Box) = -2(G''\mathcal{H} + \mathcal{H}''G)f_0 + 2G'\mathcal{H}'\mathcal{F}(\Box).\)

Trace of the previous equation is

\[\left[m^2 + (R + 3\Box)v(\Box)\right]\delta R = \mathcal{U}(\Box)\delta R = 0.\]

To solve the trace equation we use Weierstrass factorization theorem

\[\mathcal{U}(\Box)\delta R = \prod_i (\Box - \omega_i^2)e^{\gamma(\Box)}\delta R = 0,\]

where \(\omega_i^2\) are the roots of the equation \(\mathcal{U}(\omega^2) = 0\) and \(\gamma(\Box)\) is entire function. Moreover, we assume that there is no multiple roots.
Roots $\omega_i^2$ are obtained as solutions of the eigenvalue problem
\[(\Box - \omega_i^2)\delta R = 0.\]
Eigenfunctions that correspond to eigenvalue $\omega_i^2$ are denoted $\delta R_i$. General solution for $\delta R$ is the sum over all values of $\omega_i^2$ i.e. 
$\delta R = \sum_i \delta R_i$.
Eigenfunctions take the form
\[\delta R_i = (-k\tau)^{3/2} \left(C_{1i} J_{\nu_i}(-k\tau) + C_{2i} Y_{\nu_i}(-k\tau)\right),\]
where $J$, $Y$ are Bessel functions of the first and second kind respectively and $\nu_i = \sqrt{\frac{9}{4} - \frac{\omega_i^2}{H^2}}$. 
Bardeen potentials are derived from the following equations

\[-m^2(\Phi - \Psi) + v(\Box)\delta R = 0,\]
\[\delta R + (R + 3\Box)(\Phi - \Psi) = 0.\]

Then Bardeen potentials take the form

\[
\Phi + \Psi = \eta(c_1(\cos(\eta) + \eta \sin(\eta)) + c_2(-\eta \cos(\eta) + \sin(\eta))),
\]
\[
\Phi - \Psi = \frac{1}{m^2} \sum_i v(\omega_i^2)\delta R_i,
\]

where \(\eta = \frac{k\tau}{\sqrt{3}}\).
Asymptotic behavior of the Bessel function implies that Bardeen potentials are bounded if

$$\Re \nu < \frac{3}{2}.$$  

$$R - 4\Lambda + f_0 R^{p+q}(2 - p - q) = 0.$$  

This polynomial equation can be explicitly solved for $R$ if $-3 \leq p + q \leq 4$. Necessary condition for the solution to be stable is

$$1 + R^{p+q-1}(p + q)(2 - p - q)f_0 < 0.$$  

Note that if $p + q = 0$ or $p + q = 2$ there is no stable solutions. When $p + q = 1$ the stable solution might exist if $\Lambda < 0$ and $f_0 < 0$.  

Perturbations

Pervious two conditions are reformulated

\[1 - s + u = 0, \quad 1 + uz < 0,\]

where \(s = \frac{4\Lambda}{R}, z = p + q, u = f_0 R^{z-1}(2 - z).\) This system is very simple, but does not have clear physical interpretation.
References


Thank you for your attention!