# Teleparallel gravity, its modifications, and the local Lorentz invariance

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We would like to treat metric and affine structures as two different entities.

Affine connection defines the covariant derivatives

$$\bigtriangledown_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha,$$

$$\bigtriangledown_\mu A^\nu \equiv \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha.$$

One can determine the transformation properties of the connection coefficients.

$$\Gamma^{\kappa}_{\mu\nu} = \frac{\partial x^{\kappa}}{\partial x'^{\rho}} \left( \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \Gamma'^{\rho}_{\alpha\beta} + \frac{\partial^2 x'^{\rho}}{\partial x^{\mu} \partial x^{\nu}} \right).$$

Now we see that symmetric connections, $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$, are somewhat distinguished from the viewpoint of the equivalence principle since they can be set to zero at any single point by a mere coordinate transformation.
On the other hand, the antisymmetric part is a tensor, and it is known under the name of torsion

$$T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$$

.

Given an affine connection, one can generalise the notion of a straight line.

We will call a line $x^\mu(\tau)$ geodesic if an only if its tangent vector $e^\mu \equiv \frac{dx^\mu}{d\tau}$ remains tangent when parallelly transported along the line. In other words, the tangent vector is covariantly constant along the line.

Therefore, under an infinitesimal change of parameter $\tau$ we have

$$\delta \frac{dx^\mu}{d\tau} = -\Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\tau}\delta x^\nu = -\Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta\tau.$$

It gives the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\alpha} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} = 0.$$

This equation is invariant under affine changes of $\tau$.
Otherwise, there is a preferred choice of parameter, the affine parameter. Under a general non-linear reparametrisation, this equation becomes more complicated. This fact allows one to define the null infinity in general relativity. A null geodesic is infinitely long if its affine parameter reaches an infinite value.

One can ask now how to distinguish between a real curvature and a flat space in curvilinear coordinates. The answer is very simple. It is difficult to compare two vectors at a distance from each other, but it is fairly easy to do so if they are at the same point in the space. If one parallelly transports a vector along a closed contour, then the resulting vector coincides with the initial one if the space is flat. Therefore, if after such a procedure the vector has changed, it is a clear indication of the curvature.

Let us perform a parallel transport of a vector $\xi^\mu$ along a closed infinitesimal contour $\mathcal{C}$. After one revolution we have

$$\delta\xi^\mu = \oint\limits_{\mathcal{C}} d\tau \dot{\xi}^\mu$$

where the vector function $\xi^\mu(x(\tau))$ is determined by the parallel transport equation $\dot{\xi}^\mu = -\Gamma^\mu_{\nu\alpha}\xi^\alpha \frac{dx^\nu}{d\tau}$. Let us assume for simplicity that the origin of the coordinate system ($x^\mu = 0$) is chosen inside the contour, then we use the parallel transport equation and Taylor expand the vector field components and the connection coefficients around the origin

As at the lowest order it is proportional to $\oint dx^\mu = 0$, we look at the first correction:

$$\delta\xi^\mu = -\oint\limits_{\mathcal{C}} d\tau \left[\Gamma^\mu_{\nu\alpha}\xi^\alpha_{,\rho}x^\rho + \Gamma^\mu_{\nu\alpha,\rho}\xi^\alpha x^\rho + \mathcal{O}(x^2)\right]\frac{dx^\nu}{d\tau},$$

and in the first term we use the parallel transport equation again (assuming that the vector field $\xi^\mu$ is covariantly constant everywhere in vicinity of the contour), $\xi^\alpha_{,\rho} = -\Gamma^\alpha_{\rho\sigma}\xi^\sigma$:

$$\delta\xi^\mu = -\oint\limits_{\mathcal{C}} d\tau \left[-\Gamma^\mu_{\nu\alpha}\Gamma^\alpha_{\rho\sigma}\xi^\sigma x^\rho + \Gamma^\mu_{\nu\sigma,\rho}\xi^\sigma x^\rho + \mathcal{O}(x^2)\right]\frac{dx^\nu}{d\tau}.$$

Omitting the $\mathcal{O}(x^2)$-corrections, the integral is proportional to

$$\oint x^\rho \frac{dx^\nu}{d\tau} d\tau = -\oint x^\nu \frac{dx^\rho}{d\tau} d\tau = \int\int dx^\rho \wedge dx^\nu,$$

the antisymmetric area element (encircled by the contour $\mathcal{C}$) which we will denote by $S^{\rho\nu}$. The antisymmetrised coefficient must also be tensor, and finally we get

$$\delta\xi^\mu = \oint_{\mathcal{C}} d\tau \dot{\xi}^\mu = -\frac{1}{2} R^\mu{}_{\sigma\rho\nu} \xi^\sigma S^{\rho\nu}$$

where the Riemann tensor is defined as

$$R^\mu{}_{\sigma\rho\nu} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\rho\sigma,\nu} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\nu\alpha}\Gamma^\alpha_{\rho\sigma}.$$

By definition,

$$R^\mu{}_{\sigma\rho\nu} = -R^\mu{}_{\sigma\nu\rho}.$$

Of course, we could also transport a one-form instead of a vector around the contour. It yields the same tensor:

$$\delta\zeta_\mu = \oint\limits_{\mathcal{C}} d\tau \dot{\zeta}_\mu = \frac{1}{2} R^\sigma{}_{\mu\rho\nu} \zeta_\sigma S^{\rho\nu},$$

The change of sign ensures that the scalar quantities remain unchanged.

In case of **symmetric connections**, the Riemann tensor has further symmetry properties. A simple inspection of the definition readily shows that for symmetric $\Gamma$ we have

$$R^{\mu}{}_{\alpha\beta\gamma} + R^{\mu}{}_{\beta\gamma\alpha} + R^{\mu}{}_{\gamma\alpha\beta} = 0.$$

And there is also a differential identity (Bianchi identity) in this case:

$$\bigtriangledown_{\alpha} R^{\mu}{}_{\nu\beta\gamma} + \bigtriangledown_{\beta} R^{\mu}{}_{\nu\gamma\alpha} + \bigtriangledown_{\gamma} R^{\mu}{}_{\nu\alpha\beta} = 0.$$

For non-symmetric connections, we would have torsion terms in the right hand side.

There is also another way to introduce the curvature. Normally, the partial derivatives commute. However, it is not true of the covariant ones. Let us compute the commutator:

$$[\nabla_\mu \, , \, \nabla_\nu]\, \xi^\alpha = \nabla_\mu \left( \partial_\nu \xi^\alpha + \Gamma^\alpha_{\nu\beta} \xi^\beta \right) - \nabla_\nu \left( \partial_\mu \xi^\alpha + \Gamma^\alpha_{\mu\beta} \xi^\beta \right) =$$

$$= \partial_\mu \left( \partial_\nu \xi^\alpha + \Gamma^\alpha_{\nu\beta} \xi^\beta \right) + \Gamma^\alpha_{\mu\rho} \left( \partial_\nu \xi^\rho + \Gamma^\rho_{\nu\beta} \xi^\beta \right) - \Gamma^\rho_{\mu\nu} \left( \partial_\rho \xi^\alpha + \Gamma^\alpha_{\rho\beta} \xi^\beta \right) -$$

$$- \partial_\nu \left( \partial_\mu \xi^\alpha + \Gamma^\alpha_{\mu\beta} \xi^\beta \right) - \Gamma^\alpha_{\nu\rho} \left( \partial_\mu \xi^\rho + \Gamma^\rho_{\mu\beta} \xi^\beta \right) + \Gamma^\rho_{\nu\mu} \left( \partial_\rho \xi^\alpha + \Gamma^\alpha_{\rho\beta} \xi^\beta \right) =$$

$$= R^\alpha{}_{\beta\mu\nu} \xi^\beta - T^\rho{}_{\mu\nu} \nabla_\rho \, \xi^\alpha$$

where $T^\rho{}_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$ is the torsion tensor.

$$[\nabla_\mu\ ,\ \nabla_\nu]\,\xi^\alpha = R^\alpha{}_{\beta\mu\nu}\xi^\beta - T^\rho{}_{\mu\nu}\,\nabla_\rho\,\xi^\alpha$$

Alternatively, we could have used a one-form instead of a vector,

$$[\nabla_\mu\ ,\ \nabla_\nu]\,\zeta_\alpha = -R^\beta{}_{\alpha\mu\nu}\zeta_\beta - T^\rho{}_{\mu\nu}\,\nabla_\rho\,\zeta_\alpha.$$

We also have a metric structure on the manifold. The commonly adopted compatibility condition asserts that the metric tensor must be covariantly constant, $\bigtriangledown_{\mu} g_{\alpha\beta} = 0$.
This requirement,

$$\partial_{\mu} g_{\alpha\beta} = \Gamma^{\rho}_{\mu\alpha} g_{\rho\beta} + \Gamma^{\rho}_{\mu\beta} g_{\alpha\rho},$$

fixes the symmetric connection uniquely in terms of the metric

$$\Gamma^{\rho}_{\alpha\beta} = \frac{1}{2} g^{\rho\mu} \left( \partial_{\alpha} g_{\mu\beta} + \partial_{\beta} g_{\alpha\mu} - \partial_{\mu} g_{\alpha\beta} \right).$$

A general connection has also contributions from torsion and non-metricity $Q_{\mu\alpha\beta} = \bigtriangledown_{\mu} g_{\alpha\beta}$.

If we have a metric, we can raise and lower the indices. Moreover, if the connection is **metric compatible**, we can commute the metric with the covariant derivatives. It implies new symmetry properties of the Riemann tensor. We can rewrite the commutators of covariant derivatives as

$$[\bigtriangledown_\mu \ , \ \bigtriangledown_\nu]\,\xi_\alpha = R_{\alpha\beta\mu\nu}\xi^\beta,$$

$$[\bigtriangledown_\mu \ , \ \bigtriangledown_\nu]\,\zeta_\alpha = -R_{\beta\alpha\mu\nu}\zeta^\beta.$$

And we deduce:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}.$$

For Levi-Civita connection we also have

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}.$$

There is also one more respect in which the Levi-Civita connection is compatible with the metric structure. An alternative definition of the geodesic line is the path of the shortest distance between two points (provided they are not too far from each other). For the Levi-Civita connection these are the same.

In the tetrad formulation, the metric at each point is associated with the set of tangent vectors via

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$$

which defines the tetrad fields $e^a_\mu$ up to local Lorentz rotations.

Since we are interested in non-degenerate metrics, we assume that $e^a_\mu$ form a non-degenerate matrix, and inverse tetrads $e^\mu_a$ can be defined as the inverse matrix so that $e^a_\mu e^\mu_b \equiv \delta^a_b$ and $e^a_\mu e^\nu_a \equiv \delta^\nu_\mu$ with

$$g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$$

for the inverse metric.

Now we can, if we want, consider every tensor with Latin indices instead of spacetime ones with the relation between the two being understood as

$$\mathcal{T}^{a_1,\ldots,a_n}_{b_1,\ldots,b_m} \equiv e^{a_1}_{\alpha_1}\cdots e^{a_n}_{\alpha_n}\mathcal{T}^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} e^{\beta_1}_{b_1}\cdots e^{\beta_m}_{b_m}.$$

Moreover, we can now have two types of the connection coefficients, $\Gamma^{\alpha}_{\mu\beta}$ for the usual tensors and $\omega^{a}{}_{\mu b}$ for those with tangent space indices. The latter of course must transform as a connection, whilst the tetrad by construction transforms as a tensor. Therefore, under a Lorentz rotation $\Lambda$,

$$e^a_\mu \longrightarrow \Lambda^a_c e^c_\mu \,, \quad \omega^{a}{}_{\mu b} \longrightarrow \Lambda^a_c \omega^{c}{}_{\mu d} (\Lambda^{-1})^d_b - (\Lambda^{-1})^a_c \partial_\mu \Lambda^c_b \,.$$

In order to freely change the nature of the indices by the tetrads, we wish this procedure to commute with taking a covariant derivative. Obviously, this goal would be achieved by the following requirement

$$\partial_\mu e^a_\nu + \omega^a{}_{\mu b} e^b_\nu - \Gamma^\alpha_{\mu\nu} e^a_\alpha = 0$$

which can be referred to as vanishing of the "full covariant derivative" of the tetrad. With this understanding in mind, we can conveniently use tensors with indices of both types, and the covariant derivatives would be unambiguously defined for a tensor even if we are allowed to transform from one type to another.

The recipe is that we use Γ-terms for Greek indices, and ω-terms for Latin indices: $\bigtriangledown_\mu T^{a\alpha} = \partial_\mu T^{a\alpha} + \Gamma^\alpha_{\mu\beta} T^{a\beta} + \omega^a{}_{\mu b} T^{b\alpha}$.

The condition of vanishing of the "full covariant derivative" of the tetrad is solved straightforwardly to obtain

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_a \left( \partial_\mu e^a_\nu + \omega^a{}_{\mu b} e^b_\nu \right) \equiv e^\alpha_a \ \mathfrak{D}_\mu e^a_\nu$$

with $\mathfrak{D}_\mu$ being the Lorentz-covariant (with respect to the Latin index only) derivative.

or another way around

$$\omega^a{}_{\mu b} = e^a_\alpha \Gamma^\alpha_{\mu\nu} e^\nu_b - e^\nu_b \partial_\mu e^a_\nu$$

In particular, one can find the spin connection $\overset{(0)}{\omega}$ which corresponds to the Levi-Civita connection $\overset{(0)}{\Gamma}(g)$ of a given metric $g$.

Basically, both $\Gamma^{\alpha}_{\mu\beta}$ and $\omega^{a}{}_{\mu b}$ represent one and the same connection in different disguises. This conclusion is further substantiated by comparing the curvatures for both connections,

$$R^{a}{}_{b\mu\nu}(\omega) = \partial_{\mu}\omega^{a}{}_{\nu b} - \partial_{\nu}\omega^{a}{}_{\mu b} + \omega^{a}{}_{\mu c}\omega^{c}{}_{\nu b} - \omega^{a}{}_{\nu c}\omega^{c}{}_{\mu b}$$

and

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}\,,$$

which after a simple calculation gives

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = e^{\alpha}_{a} R^{a}{}_{b\mu\nu}(\omega) e^{b}_{\beta}\,.$$

In other words, the two Riemann tensors are related by mere change of the types of indices. Therefore, those are one and the same tensor under our conventions which are common for all the tensors we use.

Note also that the non-metricity in this formalism (with the vanishing of the "full covariant derivative" of the tetrad) is automatically equal to zero because

$$\nabla_\alpha g_{\mu\nu} = \eta_{ab}\left(\partial_\alpha\left(e^a_\mu e^b_\nu\right) - \Gamma^\beta_{\alpha\mu} e^a_\beta e^b_\nu - \Gamma^\beta_{\alpha\nu} e^a_\mu e^b_\beta\right)$$
$$= -e^b_\mu e^c_\nu \left(\eta_{ab}\omega^a{}_{\alpha c} + \eta_{ac}\omega^a{}_{\alpha b}\right) = 0$$

where we have used the assumption that the matrices $\omega^{\cdot}{}_{\alpha\cdot} = \omega^a{}_{\alpha b}$ belong to the Lie algebra of the group $SO(1,3)$.
In particular, absence of non-metricity leads to antisymmetry of the Riemann tensor with respect to interchange of the first two indices, $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$.

Assuming that $\bigtriangledown_\alpha g_{\mu\nu} = 0$, one can follow the standard textbook derivation of the Levi-Civita connection and prove that

$$\Gamma^\alpha_{\mu\nu} = \overset{(0)}{\Gamma}{}^\alpha_{\mu\nu}(g) + K^\alpha{}_{\mu\nu}$$

where $\overset{(0)}{\Gamma}{}^\alpha_{\mu\nu}(g)$ is the Levi-Civita connection of the metric $g$, while the tensor $K$

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\alpha\mu\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right) = \frac{1}{2}\left(T_{\mu\alpha\nu} + T_{\nu\alpha\mu} - T_{\alpha\nu\mu}\right),$$

is known under the name of contortion.
It is obviously antisymmetric with respect to two indices:

$$K_{\alpha\mu\nu} = -K_{\nu\mu\alpha}.$$

Substituting our connection into the the definition of curvature, we get

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = R^{\alpha}{}_{\beta\mu\nu}(\overset{(0)}{\Gamma}) + \overset{(0)}{\bigtriangledown}_{\mu} K^{\alpha}{}_{\nu\beta} - \overset{(0)}{\bigtriangledown}_{\nu} K^{\alpha}{}_{\mu\beta} + K^{\alpha}{}_{\mu\rho} K^{\rho}{}_{\nu\beta} - K^{\alpha}{}_{\nu\rho} K^{\rho}{}_{\mu\beta}$$

for the Riemann tensor with $\overset{(0)}{\bigtriangledown}_{\mu}$ being the covariant derivative associated to $\overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}(g)$.

Making the necessary contractions we obtain the scalar curvature

$$R(\Gamma) = R(\overset{(0)}{\Gamma}) + 2 \overset{(0)}{\bigtriangledown}_{\mu} T^{\mu} + \mathbb{T}$$

where the torsion vector is

$$T_{\mu} \equiv T^{\alpha}{}_{\mu\alpha} = -T^{\alpha}{}_{\alpha\mu},$$

and the torsion scalar can be written in several equivalent ways:

$$\begin{aligned} \mathbb{T} &= \frac{1}{2} K_{\alpha\beta\mu} T^{\beta\alpha\mu} - T_\mu T^\mu \\ &= \frac{1}{2} T_{\alpha\beta\mu} S^{\alpha\beta\mu} \\ &= \frac{1}{4} T_{\alpha\beta\mu} T^{\alpha\beta\mu} + \frac{1}{2} T_{\alpha\beta\mu} T^{\beta\alpha\mu} - T_\mu T^\mu \end{aligned}$$

with the superpotential

$$S^{\alpha\mu\nu} \equiv K^{\mu\alpha\nu} + g^{\alpha\mu} T^\nu - g^{\alpha\nu} T^\mu$$

which satisfies the antisymmetry condition

$$S^{\alpha\mu\nu} = -S^{\alpha\nu\mu}.$$

In the classical formulation of teleparallel gravity, one uses the Weitzenböck connection given by

$$\overset{\mathfrak{W}}{\omega}{}^{a}{}_{\mu b} = 0$$

or

$$\overset{\mathfrak{W}}{\Gamma}{}^{\alpha}_{\mu\nu} = e^{\alpha}_{a} \partial_{\mu} e^{a}_{\nu}$$

which is obviously curvature-free, $R^{\alpha}{}_{\beta\mu\nu}(\overset{\mathfrak{W}}{\Gamma}) = 0$.

We can denote the determinant of $e^a_\mu$ by $\|e\|$, and see from

$$R(\Gamma) = R(\overset{(0)}{\Gamma}) + 2 \overset{(0)}{\bigtriangledown}_\mu T^\mu + \mathbb{T}$$

that the action

$$S_{\mathfrak{M}\mathfrak{Y}} = - \int d^4x \|e\| \cdot \mathbb{T}$$

is equivalent to the action of GR, $\int d^4x \sqrt{-g} \cdot R(\overset{(0)}{\Gamma})$, modulo the surface term, if the Weitzenböck, or any other inertial, connection is assumed.

We are also interested in equations of motion.
We have the following first order variations for the inverse tetrad, measure, metric and torsion:

$$
\begin{aligned}
\delta e_a^{\mu} &= -e_b^{\mu} e_a^{\nu} \delta e_{\nu}^{b}, \\
\delta \|e\| &= \|e\| \cdot e_a^{\mu} \delta e_{\mu}^{a}, \\
\delta g_{\mu\nu} &= \eta_{ab} \left( e_{\mu}^{a} \delta e_{\nu}^{b} + e_{\nu}^{a} \delta e_{\mu}^{b} \right), \\
\delta g^{\mu\nu} &= -\left( g^{\mu\alpha} e_a^{\nu} + g^{\nu\alpha} e_a^{\mu} \right) \delta e_{\alpha}^{a}, \\
\delta_e T^{\alpha}{}_{\mu\nu} &= -e_a^{\alpha} T^{\beta}{}_{\mu\nu} \delta e_{\beta}^{a} + e_a^{\alpha} \left( \mathfrak{D}_{\mu} \delta e_{\nu}^{a} - \mathfrak{D}_{\nu} \delta e_{\mu}^{a} \right).
\end{aligned}
$$

In particular, for the teleparallel equivalent of GR we have $\delta_e S$

$$= -\int d^4x \|e\| \cdot \left(-2S^{\alpha\mu\nu} T_{\alpha\beta\nu} e_a^{\beta} \delta e_{\mu}^a + \mathbb{T} e_a^{\mu} \delta e_{\mu}^a - 2S_{\beta}{}^{\mu\alpha} e_a^{\beta} \mathfrak{D}_{\alpha} \delta e_{\mu}^a\right)$$

with the Lorentz-covariant derivative $\mathfrak{D}$ being equal to the ordinary one, since $\omega^b{}_{\alpha a} = 0$ in the Weitzenböck case.

We need to perform integration by parts in the last term which gives

$$2\delta e^a_\mu \cdot \left(\partial_\alpha \left(\|e\| \cdot S_\beta{}^{\mu\alpha} e^\beta_a\right) - \|e\| \cdot \omega^b{}_{\alpha a} S_\beta{}^{\mu\alpha} e^\beta_b\right)$$
$$= 2\|e\| \cdot \left(\overset{(0)}{\nabla}{}_\alpha S_\beta{}^{\mu\alpha} - K^\nu{}_{\alpha\beta} S_\nu{}^{\mu\alpha}\right) \cdot e^\beta_a \delta e^a_\mu$$

where we have used the antisymmetry of $S$ and corrected for the difference between $\Gamma$ and $\overset{(0)}{\Gamma}$ by the second term on the right hand side. Indeed, due to the antisymmetry of $S$ we have

$$\overset{(0)}{\nabla}{}_\nu S_a{}^{\mu\nu} = \frac{1}{\|e\|} \partial_\nu \left(\|e\| S^{\mu\nu}_a\right) - \overset{(0)}{\omega}{}^b{}_{\nu a} S_b{}^{\mu\nu}$$

and correct for the different connection by noting that $\omega^b{}_{\nu a} - \overset{(0)}{\omega}{}^b{}_{\nu a} = K^b{}_{\nu a}$.

Finally, using the non-degeneracy of tetrads, we get the equations of motion in the form

$$\overset{(0)}{\bigtriangledown}_{\alpha} S_{\beta}{}^{\mu\alpha} - S^{\alpha\mu\nu}\left(T_{\alpha\beta\nu} + K_{\alpha\nu\beta}\right) + \frac{1}{2}\mathbb{T}\delta^{\mu}_{\beta} = 0$$

which can be shown to be equivalent to general relativity by direct substitution of

$$R^{\alpha}{}_{\beta\mu\nu}(\overset{(0)}{\Gamma}) = -\left(\overset{(0)}{\bigtriangledown}_{\mu} K^{\alpha}{}_{\nu\beta} - \overset{(0)}{\bigtriangledown}_{\nu} K^{\alpha}{}_{\mu\beta} + K^{\alpha}{}_{\mu\rho} K^{\rho}{}_{\nu\beta} - K^{\alpha}{}_{\nu\rho} K^{\rho}{}_{\mu\beta}\right)$$

into the Einstein equation,

$$G^{\mu}_{\beta} = 0.$$

What if we covariantise the model by substituting an explicit spin connection?

Variations with respect to the spin connection coefficients can be derived exactly since

$$\delta_\omega T^{\alpha}{}_{\mu\nu} = \delta\omega^{\alpha}{}_{\mu\nu} - \delta\omega^{\alpha}{}_{\nu\mu},$$

is an exact relation for $\delta\omega^{\alpha}{}_{\mu\nu} \equiv e^{\alpha}_{a} e^{b}_{\nu} \delta\omega^{a}{}_{\mu b}$.

Suppose, we want to covariantise the teleparallel action by allowing for an arbitrary spin connection in the torsion scalar,

$$S = -\int d^4x \|e\| \cdot \mathbb{T}(e,\omega),$$

and varying independently with respect to both variables $e$ and $\omega$. We have

$$\delta_\omega S = -\int d^4x \|e\| \cdot (T^\mu{}_{\alpha\nu} + 2T_\nu \delta^\mu_\alpha)\, \delta\omega^\alpha{}_\mu{}^\nu .$$

The equation of motion is

$$T^\mu{}_{\alpha\nu} + T_\nu \delta^\mu_\alpha - T_\alpha \delta^\mu_\nu = 0$$

which (in dimension $d \neq 2$) entails $T_\mu = 0$ upon tracing, and totally

$$T^\mu{}_{\alpha\nu} = 0.$$

It does not give the desired result!

A better idea would be to vary the spin connection in the inertial class only. The latter can be imposed by demanding

$$\omega^a{}_{\mu b} = -(\Lambda^{-1})^a_c \partial_\mu \Lambda^c_b$$

where $\Lambda$ is an arbitrary Lorentz matrix and varying

$$S_{\mathfrak{W}'} = -\int d^4x \|e\| \cdot \mathbb{T}(e, \omega(\Lambda))$$

with respect to $e$ and $\Lambda$.

Literally it means that there exists a frame in which $\omega = 0$ (Weitzenböck), however one is allowed to make a local Lorentz rotation by an arbitrary matrix field $\Lambda^a_b(x)$ whose values belong to Lorentz group.

Explicit calculations are given in
*Alexey Golovnev, Tomi Koivisto, Marit Sandstad.*
*On the covariance of teleparallel gravity theories.*
*Classical and Quantum Gravity 34 (2017) 145013*
*https://arxiv.org/abs/1701.06271*

However, the essence is very simple. Varying the spin connection with fixed tetrads does not change the Levi-Civita connection, while we know that in any case

$$\delta_\Lambda \mathbb{T} = \delta_\Lambda R(\omega) - 2 \overset{(0)}{\nabla}_\mu (\delta_\Lambda T^\mu)$$

where $\delta_\Lambda(...) = \delta_\omega(...) \cdot \delta_\Lambda \omega$.
Since $R(\omega(\Lambda)) \equiv 0$, the variation $\delta_\omega S_{\mathfrak{W}'}$ is a surface term and does not produce any new equation of motion. The model, though locally Lorentz covariant, is then equivalent to teleparallel gravity.

Equivalently, one could also impose ${R^a}_{b\mu\nu} = 0$ with a Lagrange multiplier. Then the action would be

$$S_{\mathfrak{LW}} = -\int d^4x \|e\| \cdot \left(\mathbb{T}(e,\omega) + {\lambda_a}^{b\mu\nu} {R^a}_{b\mu\nu}(\omega)\right)$$

where ${\lambda_a}^{b\mu\nu}$ is a Lagrange multiplier with the symmetry properties $\lambda^{ab\mu\nu} = -\lambda^{ab\nu\mu}$ and $\lambda^{ab\mu\nu} = -\lambda^{ba\mu\nu}$.

Finally, we would also like to notice that if we take an action

$$S_{\omega-free} = -\int d^4x \|e\| \left( \mathbb{T} - e_a^\mu R^a{}_{b\mu\nu}(\omega) e_c^\nu \eta^{bc} \right),$$

then it is equivalent to the Einstein-Hilbert action modulo a surface term, with an arbitrary spin connection, not necessarily inertial.
The spin connection makes only a superficial appearance here, and one can formulate equations of general relativity in terms of arbitrary spin connection with geometries containing both curvature and torsion.
Unfortunately, a quick look at variations shows that a generalisation with non-linear functions of this Lagrangian density does not give a non-trivial model.

It would be interesting to try making modifications of GR in the teleparallel framework. One very popular example is $f(T)$ gravity.

The covariantisation procedure works differently in generalised teleparallel gravities. Since the dependence on the spin connection in generalised models cannot be reduced to a surface term, the variation $\delta_\omega$ produces non-trivial equations of motion.

However, admissible variations of $\omega$ in the inertial class amount to local Lorentz transformations. Note also that a covariantised action is, by definition, identically invariant under simultaneous local Lorentz transformation of the spin connection and the tetrad (and other non-trivially transforming fields if there are some). Therefore, the stationarity of the action under local Lorentz transformations of the spin connection is equivalent to that under local Lorentz rotations of tetrads. The latter is already ensured given that the equations of motion for the tetrad are satisfied since the local Lorentz rotation is nothing but a special class of variations of the tetrad.

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Let us consider an $f(T)$ model with inertial spin connection,

$$S_{f(T)} = -\int d^4x \|e\| \cdot f\left(\mathbb{T}(e, \omega(\Lambda))\right).$$

We want to derive equations of motion.

Making the variation with respect to the tetrad gives

$$f'(\mathbb{T}) \cdot \overset{(0)}{R}{}^{\mu\nu} + K^{\mu\nu\alpha}\partial_\alpha f'(\mathbb{T}) - T^\mu \partial^\nu f'(\mathbb{T}) + \frac{1}{2} f(\mathbb{T}) \cdot g^{\mu\nu} = 0$$

Unlike in the TEGR case, this equation has a non-trivial antisymmetric part

$$T^{\alpha\mu\nu}\partial_\alpha f'(\mathbb{T}) + T^\nu \partial^\mu f'(\mathbb{T}) - T^\mu \partial^\nu f'(\mathbb{T}) = 0$$

which reflects the non-invariance under local Lorentz rotations of tetrads.

Variation with respect to the purely inertial spin connection gives the same result.

We have seen that there are several methods of approaching teleparallel gravity covariantly. They can be classified according to how the variation of the spin connection is performed.

**0.** "Weitzenböck variation": $\omega = 0$. This is the historical formulation of teleparallel gravity, and it is not covariant.

**1.** "Fixed omega variation": $\omega$ is totally fixed but arbitrary. It can be thought of as writing the Weitzenböck action in an arbitary frame by making a local Lorentz rotation. Under any fixed choice, the Lagrangian is not invariant, however there is the freedom of making this choice.

**2.** "Independent variation" of the unrestricted $\omega$. It results in no gravity at all, $T = 0$.

**3.** "Inertial variation": $\omega$ is varied in the class of inertial spin connections. Independent variables are tetrads and Lorentz matrices which parametrise the spin connection. This is a proper covariantisation for both teleparallel equivalent of GR and modified teleparallel models.
**4.** "Constrained variation": $\omega$ is by itself an independent variable, however its curvature tensor is set to zero with a Lagrange multiplier. This is equivalent to previous option plus an equation for the Lagrange multiplier.
**5.** "Compensating variation": decoupled $\omega$ with $\mathbb{T} - e_a^{\mu} R^a{}_{b\mu\nu}(\omega) e_c^{\nu} \eta^{bc}$ for the Lagrangian density. Strictly speaking, it is no longer a teleparallel model. The spin connection is arbitrary indeed, and gravity can be expressed as a combination of torsion and curvature, in any proportions we like. It works for a teleparallel equivalent of GR. Whether it can be used for modified gravity scenarios, remains to be seen.

Thank you for your attention!