A Bosonization of $U_q(\widehat{sl}(M|N))$

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Summary

- We obtain a bosonization of $U_q(\widehat{sl}(M|N))$ for an arbitrary level $k \in \mathbb{C}$
- We propose a bosonization of the vertex operator for $k \neq -M+N$.

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Bosonization

- Realization by differential operators

$$\left[\frac{\partial}{\partial x}, x\right] = \frac{\partial}{\partial x}\cdot x - x\cdot\frac{\partial}{\partial x} = 1$$

- We call " $x, \frac{\partial}{\partial x}$ " Boson.

Bosonization

Bosonization is a powerful method to study

(1) representation theory

ex. 1. q-Virasoro

and

(2) its application to mathematical physics.

ex. XXZ chain

Bosonization is useful

Example 1 : XXZ chain

- **Hamiltonian**

$$\left(\Delta = \frac{q + q^{-1}}{2} \right)$$

$$H_{XXZ} = -\frac{1}{2} \sum_{k \in \mathbb{Z}} \left(\sigma_k^x \sigma_{k+1}^x + \sigma^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \right).$$

$$H_{xxz} \curvearrowright \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots$$

- **Symmetry**

$$U_q \widehat{sl}(2)$$

Correlation function

$$\langle \sigma_1^z \rangle = \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1+q^{2n}} \right)^2$$ [Baxter, (1973)]

n-point correlation function

$$\langle (E_{\varepsilon_n' \varepsilon_n})_n \cdots (E_{\varepsilon_2' \varepsilon_2})_2 (E_{\varepsilon_1' \varepsilon_1})_1 \rangle \ ?$$

We use a bosonization of $U_q(\widehat{sl}(2))$.

n-point correlation function

XXZ chain $U_q(\widehat{sl}(2))$

$$\left\langle (E_{\varepsilon_n'\varepsilon_n})_n \cdots (E_{\varepsilon_2'\varepsilon_2})_2 (E_{\varepsilon_1'\varepsilon_1})_1 \right\rangle$$

$$= \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty} \prod_{a\in A} \oint_C \frac{dw_a}{(w_a-1)} \prod_{a'\in A'} \oint_{C'} \frac{dw'_{a'}}{(w'_{a'}-1)} \times$$

$$\times \prod_{\substack{a\le j\le n \\ a\in A}} \frac{w_a-q^2}{w_a-1} \prod_{\substack{a'<j\le n \\ a'\in A'}} \frac{1-q^2 w'_{a'}}{1-w'_{a'}} \prod_{j<k} \frac{\eta_k-\eta_j}{\eta_k-q^2\eta_j}$$

$$\times \frac{\prod_{j<k} h(\eta_j/\eta_k)}{\prod_j h(\eta_j)^n} \times \Theta_{q^4}\left(-q^2 \prod_j \eta_j^{-2}\right).$$

$$(z;p)_\infty = \prod_{n=0}^{\infty} (1-p^n z), \quad \Theta_p(z) = (p;p)_\infty (z;p)_\infty (p/z;p)_\infty$$

$$h(z) = (q^2 z;q^2)_\infty (q^2/z;q^2)_\infty$$

Example 2 : q-Virasoro algebra

Virasoro

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\, \delta_{m+n,0}$$

q-Virasoro

$$[T_m, T_n] = -\sum_{\ell=1}^{\infty} f_\ell \left(T_{m-\ell} T_{n+\ell} - T_{n-\ell} T_{m+\ell} \right) - \frac{(1-q)(1-t^{-1})}{1-p} (p^m - p^{-m})\, \delta_{m+n,0}$$

- $p = q/t$
- $f(z) = 1 + \sum_{\ell=1}^{\infty} f_\ell z^\ell = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{1+p^n} z^n \right)$

- $t = q^{\beta}$, $p = q^{1-\beta}$, $q = e^{h}$ $(h \to 0)$

$$T_n = 2\delta_{n,0} + \beta\left(L_n + \frac{(1-\beta)^2}{4\beta} + \delta_{n,0}\right)h^2 + \cdots$$

- central charge

$$c = 1 - \frac{6(1-\beta)^2}{\beta}$$

q-Virasoro algebra

algebra	Virasoro	q-Virasoro
Singular Vector	$\lvert \chi_{r,s} \rangle \sim J_{(s^r)}(x; 1/\beta)$ **Jack Polynomial** of boson	$\lvert \chi_{r,s} \rangle \sim J_{(s^r)}(x; q, t)$ **Macdonald Poly.** of boson

- Singular Vector $\lvert \chi \rangle$

  $L_n \lvert \chi \rangle = 0 \quad (n > 0), \quad L_0 \lvert \chi \rangle = s \lvert \chi \rangle$

Boson

$$[a_m, a_n] = m \frac{1-q^{|m|}}{1-t^{|m|}} \delta_{m+n,0}$$

Bosonization

- $$T(z) = \sum_{m \in \mathbb{Z}} T_n z^{-n} = \Lambda^{+}(p^{-\frac{1}{2}} z) + \Lambda^{-}(p^{\frac{1}{2}} z)$$

- $$\Lambda^{\pm}(z) = \exp\left(\pm \sum_{n>0} \frac{1-t^{-n}}{1+p^{n}} \frac{a_{-n}}{n} z^{n} \right) \times \exp\left(\mp \sum_{n>0} (1-t^{n}) \frac{a_n}{n} z^{-n} \right) p^{\pm \frac{1}{2}} q^{\pm \beta a_0}$$

[Shiraish, Kubo, Awata, Odake (1996)]

Bosonization

- Bosonization is a realization of differential operators.
- Bosonization is a powerful method to study representation theory and its application to mathematical physics.

Outline

Part I Review of Bosonizations

$U_q(\widehat{sl}(2))$, $U_q(\widehat{sl}(3))$

Part II Recent Progress

§1 Bosonization of $U_q(\widehat{sl}(M|N))$

§2 Bosonization of Screenings

§3 Bosonization of Vertex Operators

Part I Review of Bosonizations

$U_q(\widehat{sl}(2))$, $U_q(\widehat{sl}(3))$

Quantum algebra $U_q(\widehat{sl}(2))$

Generators $a_n, x_m^{\pm}, h, k \quad (n \in \mathbb{Z}_{\neq 0}, m \in \mathbb{Z})$

Relations

- $$[a_m, a_n] = \frac{[2m]_q [km]_q}{m} \delta_{m+n,0}$$

- $$(z_1 - q^{\pm 2} z_2) X^{\pm}(z_1) X^{\pm}(z_2) = (q^{\pm 2} z_1 - z_2) X^{\pm}(z_2) X^{\pm}(z_1)$$

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad X^{\pm}(z) = \sum_{m \in \mathbb{Z}} x_m^{\pm} z^{-m-1}$$

Quantum algebra $U_q(\widehat{sl}(2))$

- $[X^+(z_1), X^-(z_2)] = \dfrac{1}{(q-q^{-1})z_1 z_2} \times$

$$\times \left(\delta\left(q^{k}\frac{z_2}{z_1}\right)\psi^+(q^{\frac{k}{2}}z_2) - \delta\left(q^{-k}\frac{z_2}{z_1}\right)\psi^-(q^{-\frac{k}{2}}z_2) \right)$$

- $\psi^{\pm}(q^{\pm\frac{k}{2}}z) = q^{\pm h} \exp\left(\pm(q-q^{-1})\sum_{m>0} a_{\pm m} z^{\mp m}\right)$

- $[h, X^{\pm}(z)] = \pm 2 X^{\pm}(z)$

- $[k, U_q(\widehat{sl}(2))] = 0$ Center

$$\delta(z) = \sum_{m\in\mathbb{Z}} z^m$$

Realization of $U_q(\widehat{sl}(2))$

k : Level $\qquad [k, U_q\widehat{sl}(2)] = 0$

① $k = 0$ finite dimensional representation

② $k = 1$ [Frenkel, Jing : Proc. Nat. Acad. Scie. (1988)]

③ $k \in \mathbb{C}$ [Shiraishi : Phys. Lett. A (1992)]
[Matsuo : CMP (1994)]

③ is completely different from ②.

Level-1 Bosonization

$U_q(\widehat{sl}(2)),\ k=1$

$$X^{\pm}(z) = \exp\left(\pm \sum_{m>0} \frac{a_{-m}}{[m]_q} q^{\mp\frac{m}{2}} z^{m}\right) \times \exp\left(\mp \sum_{m>0} \frac{a_m}{[m]_q} q^{\mp\frac{m}{2}} z^{-m}\right) e^{\alpha} z^{\pm\partial}$$

- a_m : generators of $U_q(\widehat{sl}(2))$

  $$[a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n,0}$$

- $[\partial, \alpha] = 2$: zero mode operator

Normal Ordering $: \; :$

$$:a_m a_{-n}: = a_{-n} a_m$$

$$:a_{-n} a_m: = a_{-n} a_m \qquad (m, n > 0)$$

$$:\alpha \cdot \partial: = \alpha \cdot \partial$$

Example

$$X^{\pm}(z) = :\exp\left(\mp \sum_{m \neq 0} \frac{a_m}{[m]_q} q^{\mp\frac{m}{2}} z^{-m} + \alpha \pm \partial\right):$$

Level-1 Bosonization $U_q(\widehat{sl}(3))$

$$X_j^{\pm}(z) = :\exp\left(\mp \sum_{m\neq 0} \frac{a_m^j}{[m]_q} q^{\mp\frac{m}{2}} z^{-m} + \alpha_j \pm \partial_j\right):$$

$$h_j = \partial_j \qquad (j=1,2)$$

- a_m^i = generators of $U_q(\widehat{sl}(3))$

$$[a_m^i, a_n^j] = \frac{[A_{ij}m]_q [m]_q}{m} \delta_{m+n,0}$$

- $[\partial_i, \alpha_j] = A_{ij}$: zero mode operators

[Frenkel, Jing : Proc. Nat. Acad. Scie. (1988)]

Quantum algebra $U_q(\widehat{sl}(3))$

Higher-Rank

Generators

$a_m^i \quad (i=1,2 \,;\, m \in \mathbb{Z}_{\neq 0})$

$x_m^{i\pm} \quad (i=1,2 \,;\, m \in \mathbb{Z})$

$h_i, \, k \quad (i=1,2)$

Relations

$$[a_m^i, a_n^j] = \frac{[A_{ij} m]_q [k m]_q}{m} \delta_{m+n,0}$$

$$(A_{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{Cartan matrix}$$

Quantum algebra $U_q(\widehat{sl}(3))$

- $[a_m^i, X_j^{\pm}(z)] = \pm \frac{[A_{ij} m]_q}{m} q^{\mp \frac{|m|}{2}} z^m X_j^{\pm}(z)$

- $(z_1 - q^{\pm A_{ij}} z_2) X_i^{\pm}(z_1) X_j^{\pm}(z_2)$
$$= (q^{\pm A_{ij}} z_1 - z_2) X_j^{\pm}(z_2) X_i^{\pm}(z_1)$$

- $[X_i^{+}(z_1), X_j^{-}(z_2)] = \frac{1}{(q-q^{-1}) z_1 z_2} \times$
$$\times \left(\delta\left(q^{k} \frac{z_2}{z_1}\right) \psi_i^{+}(q^{\frac{k}{2}} z_2) - \delta\left(q^{-k} \frac{z_2}{z_1}\right) \psi_i^{-}(q^{-\frac{k}{2}} z_2) \right)$$

- $[k, U_q(\widehat{sl}(3))] = 0$

Quantum algebra $U_q(\widehat{sl}(3))$

- $[X_i^{\pm}(z_1), X_j^{\pm}(z_2)] = 0$ for $|A_{ij}| = 0$

- $$\{X_i^{\pm}(z_1)X_i^{\pm}(z_2)X_j^{\pm}(z) - (q+q^{-1})X_i^{\pm}(z_1)X_j^{\pm}(z)X_i^{\pm}(z_2) + X_j^{\pm}(z)X_i^{\pm}(z_1)X_i^{\pm}(z_2)\} + \{z_1 \Leftrightarrow z_2\} = 0$$

  for $|A_{ij}| = 1$

Serre relation

Level-k Bosonization

$U_q(\widehat{sl}(2))$, $k \in \mathbb{C}$

Boson $\alpha_m, \beta_m, \gamma_m, Q_\beta, Q_\gamma$

$$[\alpha_m, \alpha_n] = \frac{[(k+2)m]_q [2m]_q}{m} \delta_{m+n,0}$$

$$[\beta_m, \beta_n] = -\frac{[m]_q^2}{m} \delta_{m+n,0}$$

$$[\gamma_m, \gamma_n] = \frac{[m]_q^2}{m} \delta_{m+n,0}$$

$$[\beta_0, Q_\beta] = -1, \quad [\gamma_0, Q_\gamma] = 1$$

Level - k Bosonization

$U_q(\widehat{sl}(2))$

$$\beta(z) = -\sum_{m\neq 0} \frac{\beta_m}{[m]_q} z^{-m} + Q_\beta + \beta_0 \log z$$

$$\gamma(z) = -\sum_{m\neq 0} \frac{\gamma_m}{[m]_q} z^{-m} + Q_\gamma + \gamma_0 \log z$$

$$\beta_\pm(z) = \pm (q - q^{-1}) \sum_{\pm m > 0} \beta_m z^{-m} \pm \beta_0 \log q$$

$$\alpha_\pm(z) = \pm (q - q^{-1}) \sum_{\pm m > 0} \alpha_m z^{-m} \pm \alpha_0 \log q$$

Level-k Bosonization $U_q(\widehat{sl}(2))$

- $$X^+(z) = \frac{1}{(q-q^{-1})z}\Big\{ :\exp\big(B_+(z) - (B+\gamma)(qz)\big): - :\exp\big(B_-(z) - (B+\gamma)(q^{-1}z)\big): \Big\}$$

- $$X^-(z) = \frac{1}{(q-q^{-1})z}\Big\{ :\exp\big(\alpha_-(q^{-\frac{k+2}{2}}z) + B_-(q^{-k-2}z) + (B+\gamma)(q^{-k-1}z)\big): - :\exp\big(\alpha_+(q^{\frac{k+2}{2}}z) + B_+(q^{k+2}z) + (B+\gamma)(q^{k+1}z)\big): \Big\}$$

- $$a_m = q^{-\frac{k+2}{2}|m|}\alpha_m + (q^{-k|m|} + q^{-(k+2)|m|})B_m$$

Level-k Bosonization $\quad U_q(\widehat{sl}(3))$

Boson $\quad \alpha_m^{i} \ (i=1,2), \ \beta_m^{ij}, \ \gamma_m^{ij} \quad (1 \leq i < j \leq 3)$

- $$[\alpha_m^{i}, \alpha_n^{j}] = \frac{[(k+3)m]_q [A_{ij} m]_q}{m} \delta_{m+n,0}$$
- $$[\beta_m^{ij}, \beta_n^{i'j'}] = -\frac{[m]_q^2}{m} \delta_{ii'} \delta_{jj'} \delta_{m+n,0}$$
- $$[\gamma_m^{ij}, \gamma_n^{i'j'}] = +\frac{[m]_q^2}{m} \delta_{ii'} \delta_{jj'} \delta_{m+n,0}$$

- $$[b_0^{ij}, Q_b^{i'j'}] = -\delta_{ii'} \delta_{jj'}$$
- $$[c_0^{ij}, Q_c^{i'j'}] = \delta_{ii'} \delta_{jj'}$$

Level - k Bosonization $\quad U_q(\widehat{sl}(3))$

$$\beta^{\bar{i}\bar{j}}(z) = -\sum_{m\neq 0} \frac{\beta_m^{\bar{i}\bar{j}}}{[m]_q} z^{-m} + Q_\beta^{\bar{i}\bar{j}} + \beta_0^{\bar{i}\bar{j}} \log z$$

$$\gamma^{\bar{i}\bar{j}}(z) = -\sum_{m\neq 0} \frac{\gamma_m^{\bar{i}\bar{j}}}{[m]_q} z^{-m} + Q_\gamma^{\bar{i}\bar{j}} + \gamma_0^{\bar{i}\bar{j}} \log z$$

$$\beta_\pm^{\bar{i}\bar{j}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} \beta_m^{\bar{i}\bar{j}} z^{-m} \pm \beta_0^{\bar{i}\bar{j}} \log q$$

$$\alpha_\pm^{\bar{i}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} \alpha_m^{\bar{i}} z^{-m} \pm \alpha_0^{\bar{i}} \log q$$

Level-k Bosonization $\qquad U_q(\widehat{sl}(3))$

- $$X_1^+(z) = \frac{1}{(q-q^{-1})z}\left\{ :\exp\left(B_+^{12}(z) - (B+\partial)^{12}(qz)\right): - :\exp\left(B_-^{12}(z) - (B+\partial)^{12}(q^{-1}z)\right): \right\}$$

- $$X_2^+(z) = \frac{1}{(q-q^{-1})z}\left\{ :\exp\left((B+\partial)^{12}(z) + B_+^{13}(z) - (B+\partial)^{13}(qz)\right): - :\exp\left((B+\partial)^{12}(z) + B_-^{13}(z) - (B+\partial)^{13}(q^{-1}z)\right): \right\}$$
$$+ \frac{1}{(q-q^{-1})z}\left\{ :\exp\left(B_+^{23}(qz) - (B+\partial)^{23}(q^2z) + B_+^{13}(z) - B_+^{12}(qz)\right): - :\exp\left(B_-^{23}(qz) - (B+\partial)^{23}(z) + B_+^{13}(z) - B_+^{12}(qz)\right): \right\}$$

Level-k Bosonization $\quad U_q(\widehat{sl}(3))$

$$
\begin{aligned}
\bullet\, X_1^-(z) = \frac{1}{(q-q^{-1})z} \Big\{ &: \exp\Big(B_+^{12}(q^{k+2}z) + (B+r)^{12}(q^{k+1}z) + \alpha_+^1(q^{\frac{k+3}{2}}z) \\
&\qquad + B_+^{13}(q^{k+3}z) - B_+^{23}(q^{k+2}z) \Big) \\
&- \exp\Big(B_-^{12}(q^{-k-2}z) + (B+r)^{12}(q^{-k-1}z) + \alpha_-^1(q^{-\frac{k+3}{2}}z) \\
&\qquad + B_-^{13}(q^{-k-3}z) - B_-^{23}(q^{-k-2}z) \Big) : \Big\} \\
+ \frac{1}{(q-q^{-1})z} \Big[&: \Big\{ \exp\Big(B_+^{23}(q^{k+2}z) - (B+r)^{23}(q^{k+3}z) \Big) \\
&- \exp\Big(B_-^{23}(q^{k+2}z) - (B+r)^{23}(q^{k+1}z) \Big) \Big\} \\
&\times \exp\Big(\alpha_+^1(q^{\frac{k+3}{2}}z) + (B+r)^{13}(q^{k+2}z) + B_+^{13}(q^{k+3}z) - B_+^{23}(q^{k+2}z) \Big) : \Big]
\end{aligned}
$$

Level-k Bosonization $U_q(\widehat{sl}(3))$

$$\bullet\ X_2^-(z)=\frac{1}{(q-q^{-1})z}:\Big\{\exp\Big(-B_+^{12}(q^{-k-1}z)-(B+\gamma)^{12}(q^{-k-2}z)\Big)$$

$$-\exp\Big(-B_-^{12}(q^{-k-1}z)-(B+\gamma)^{12}(q^{-k}z)\Big)\Big\}$$

$$\times\exp\Big(\alpha_-^{2}(q^{-\frac{k+3}{2}}z)+(B+\gamma)^{13}(q^{-k-1}z)+B_-^{23}(q^{-k-1}z)+B_-^{23}(q^{-k-3}z)\Big):\Big]$$

$$+\frac{1}{(q-q^{-1})z}\Big\{:\exp\Big(B_+^{23}(q^{k+3}z)+(B+\gamma)^{23}(q^{k+2}z)+\alpha_+^{2}(q^{\frac{k+3}{2}}z)\Big)$$

$$-\exp\Big(B_-^{23}(q^{-k-3}z)+(B+\gamma)^{23}(q^{-k-2}z)+\alpha_-^{2}(q^{-\frac{k+3}{2}}z)\Big):\Big\}$$

[Awata, Odake, Shiraishi : CMP (1994)]

How many sums ?

An interpretation based on flag manifold is given by

[Feigin, Frenkel : CMP (1990)]

$U_q(\widehat{sl}(2))$

	$X^+(z)$	$X^-(z)$
$k=1$	1	1
$k\in\mathbb{C}$	2	2

$U_q(\widehat{sl}(3))$

	$X_1^+(z)$	$X_2^+(z)$	$X_1^-(z)$	$X_2^-(z)$
$k=1$	1	1	1	1
$k\in\mathbb{C}$	2	4	4	4

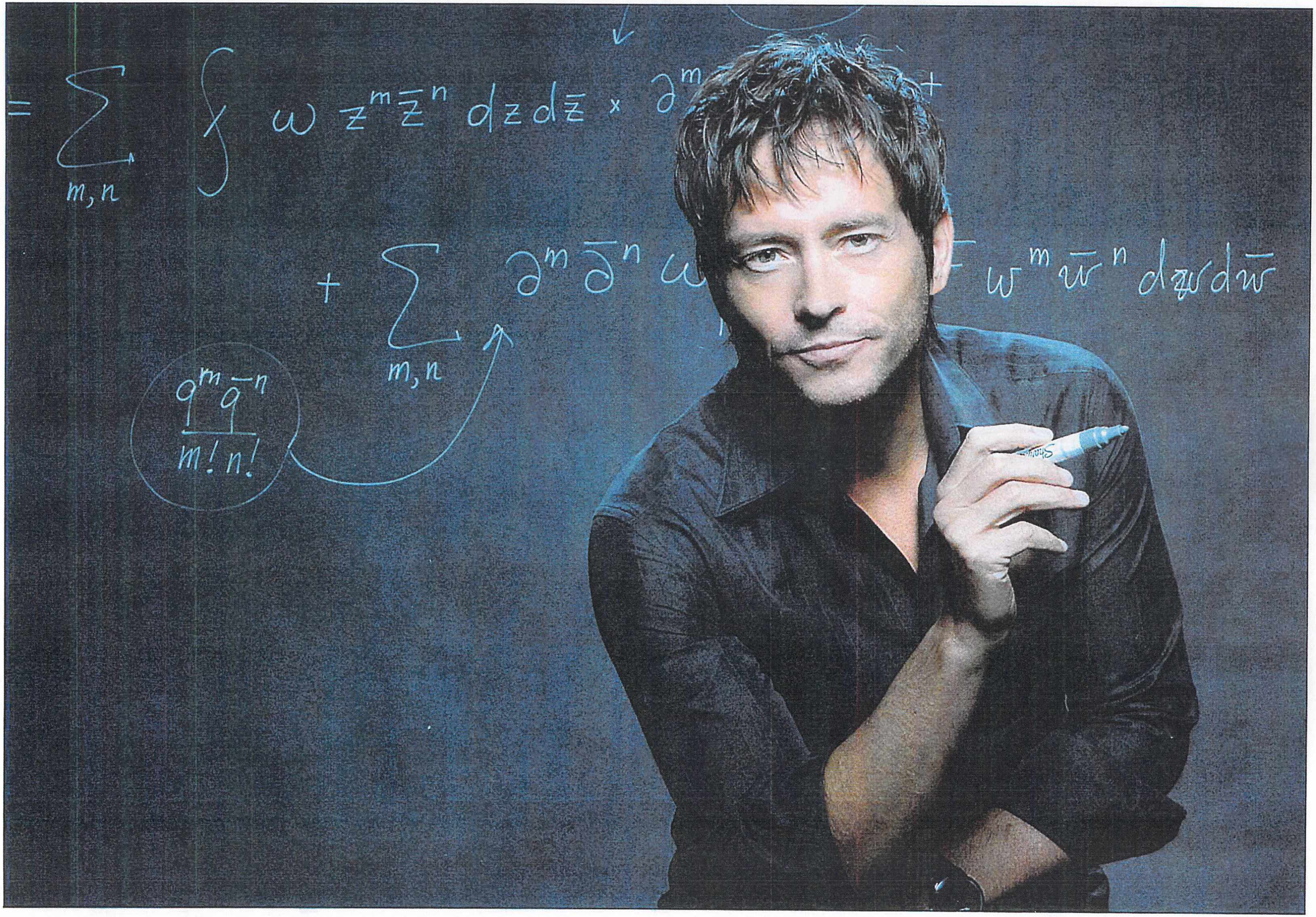
Love and Math by E. Frenkel

$\widehat{sl}_N$

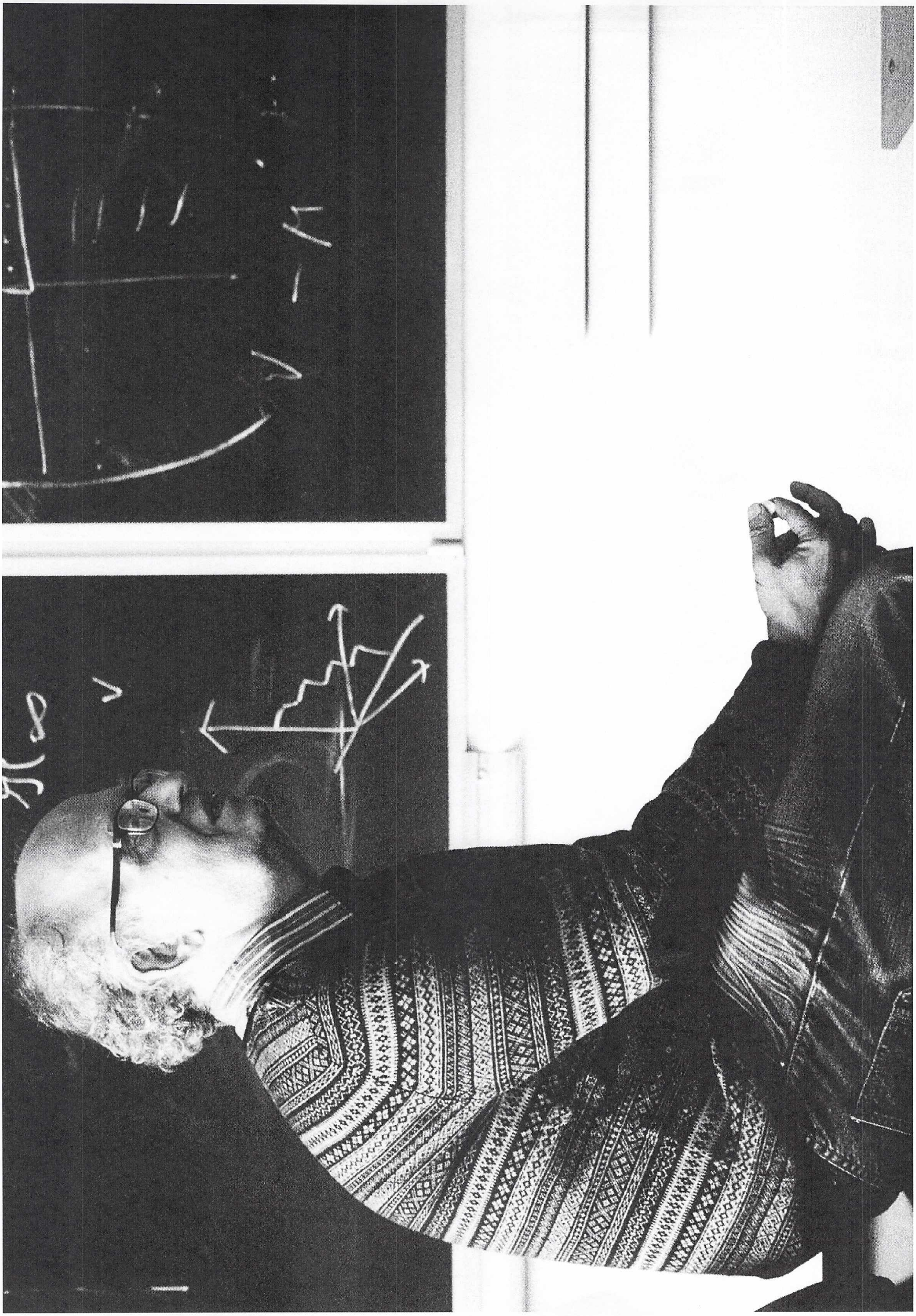
$U_q(\widehat{sl}(N))$

	$X_j^+(z)$	$X_j^-(z)$
$k=1$	1	1
$k\in\mathbb{C}$	$2j$	$2(N-1)$

$(j=1,2,\cdots,N-1)$



$= \sum_{m,n} \int \omega z^m \bar{z}^n dz d\bar{z} \times \partial^m$
$+ \sum_{m,n} \partial^m \bar{\partial}^n \omega$
$w^m \bar{w}^n dw d\bar{w}$
$\frac{q^m \bar{q}^n}{m!n!}$



Quantum algebra $U_q(\widehat{sl}(3))$

- $[a_m^i, X_j^{\pm}(z)] = \pm \frac{[A_{ij} m]_q}{m} q^{\mp \frac{|m|}{2}} z^m X_j^{\pm}(z)$

- $(z_1 - q^{\pm A_{ij}} z_2) X_i^{\pm}(z_1) X_j^{\pm}(z_2)$
  $= (q^{\pm A_{ij}} z_1 - z_2) X_j^{\pm}(z_2) X_i^{\pm}(z_1)$

- $[X_i^{+}(z_1), X_j^{-}(z_2)] = \frac{1}{(q-q^{-1}) z_1 z_2} \times$
  $\times \left(\delta\left(q^{k} \frac{z_2}{z_1}\right) \psi_i^{+}(q^{\frac{k}{2}} z_2) - \delta\left(q^{-k} \frac{z_2}{z_1}\right) \psi_i^{-}(q^{-\frac{k}{2}} z_2) \right)$

- $[k, U_q(\widehat{sl}(3))] = 0$

$$:\exp\left(\mathcal{B}_\pm(z) \pm (\mathcal{B}+\gamma)(q^{\mp 1}z)\right):$$

$$\left[:e^{\mathcal{B}_+(z_1) - (\mathcal{B}+\gamma)(qz_1)}: , :e^{\mathcal{B}_+(z_2) + (\mathcal{B}+\gamma)(q^{-1}z_2)}: \right]$$

$$= -(q-q^{-1})\,\delta\left(q^{-2}\frac{z_2}{z_1}\right) :e^{\mathcal{B}_+(z_1)+\mathcal{B}_+(z_2)}: ,$$

$$\left[:e^{\mathcal{B}_-(z_1) - (\mathcal{B}+\gamma)(q^{-1}z_1)}: , :e^{\mathcal{B}_-(z_2) + (\mathcal{B}+\gamma)(qz_2)}: \right]$$

$$= (q-q^{-1})\,\delta\left(q^{2}\frac{z_2}{z_1}\right) :e^{\mathcal{B}_-(z_1)+\mathcal{B}_-(z_2)}: .$$

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n$$

Level-k Bosonization　　$U_q(\widehat{sl}(3))$

$$\bullet\ X_1^-(z)=\frac{1}{(q-q^{-1})z}\Big\{:\exp\Big(B_+^{12}(q^{k+2}z)+(B+r)^{12}(q^{k+1}z)+\alpha_+^1(q^{\frac{k+3}{2}}z)$$

$$+B_+^{13}(q^{k+3}z)-B_+^{23}(q^{k+2}z)\Big)$$

$$-\exp\Big(B_-^{12}(q^{-k-2}z)+(B+r)^{12}(q^{-k-1}z)+\alpha_-^1(q^{-\frac{k+3}{2}}z)$$

$$+B_-^{13}(q^{-k-3}z)-B_-^{23}(q^{-k-2}z)\Big):\Big\}$$

$$+\frac{1}{(q-q^{-1})z}\Big[:\Big\{\exp\Big(B_+^{23}(q^{k+2}z)-(B+r)^{23}(q^{k+3}z)\Big)$$

$$-\exp\Big(B_-^{23}(q^{k+2}z)-(B+r)^{23}(q^{k+1}z)\Big)\Big\}$$

$$\times\exp\Big(\alpha_+^1(q^{\frac{k+3}{2}}z)+(B+r)^{13}(q^{k+2}z)+B_+^{13}(q^{k+3}z)-B_+^{23}(q^{k+2}z)\Big):\Big]$$

Level-k Bosonization

Wakimoto realization

- $\widehat{sl}(2)$ [Wakimoto ; CMP (1986)]
- $\widehat{sl}(N)$ [Feigin-Frenkel ; Phys. Math. Strings (1990)]
- $U_q\widehat{sl}(2)$ [Matsuo ; CMP (1994)]
  [Shiraishi ; Phys. Lett. A (1992)]
- $U_q\widehat{sl}(N)$ [Awata-Odake-Shiraishi ; CMP (1994)]
- $U_q\widehat{sl}(M|N)$ [Kojima ; CMP (2017)]

Summary of Part I

- We studied bosonizations of $U_q(\widehat{sl}(N))$.
- Bosonizations for level $k \in \mathbb{C}$ is completely different from those for $k=1$.
- Higher-rank generalization is nontrivial for level $k \in \mathbb{C}$.

Part II : Recent Progress

§1 A bosonization of $U_q(\widehat{sl}(M|N))$

Table of Bosonization

$U_q(\mathfrak{g})$

Level $k=1$ — q-Frenkel-Kac realization

$\mathfrak{g} = (ADE)^{(r)}, (BC)^{(1)}, G_2^{(1)}, \widehat{sl}(M|N), \widehat{osp}(2|2)^{(2)}$

$(r=1,2)$

Level $k \in \mathbb{C}$ — q-Wakimoto realization

$\mathfrak{g} = \widehat{sl}(N)$, $\widehat{sl}(M|N)$

$\widehat{sl}(2|1)$ [Awata-Odake-Shiraishi, LMP42 (1997)]

[Yang-Zhang-Liu, JMP 48 (2002)]

Superalgebra $U_q(\widehat{sl}(M|N))$

$sl(M|N)$ [Kac : Adv. Math. (1977)]

$\widehat{sl}(M|N)$ [Van. de. Leur: CMP (1989)]

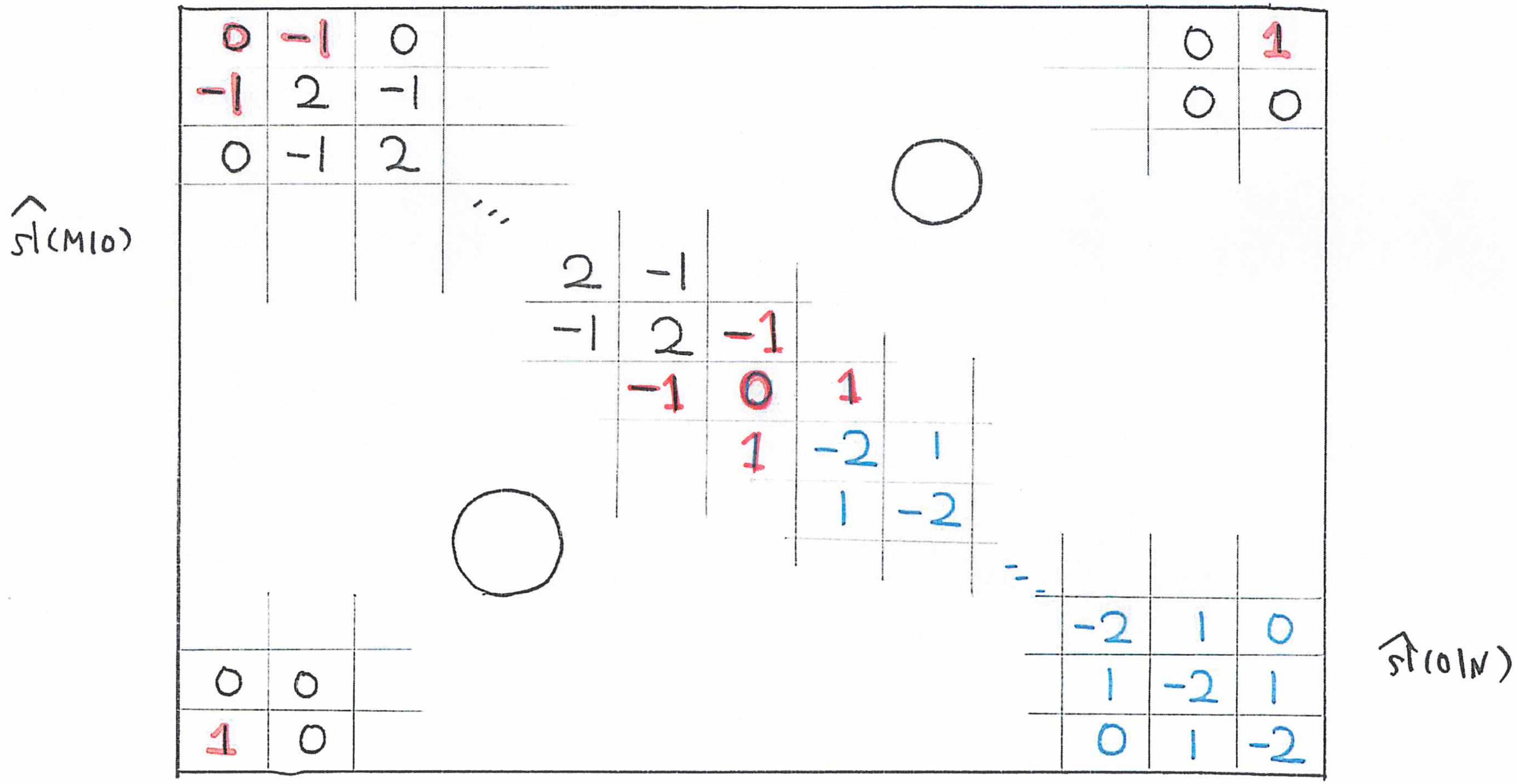
$U_q(\widehat{sl}(M|N))$ [Yamane : Publ. RIMS (1999)]

$$[x, y] = xy - yx$$

$$\{x, y\} = xy + yx$$ ← fermion

Superalgebra $U_q(\widehat{sl}(M|N))$ [Yamane : RIMS 35 (1999)]

Cartan matrix $(A_{ij})_{0 \leq i,j \leq M+N-1}$



Superalgebra $U_q(\widehat{sl}(M|N))$

Generators

$$X^{\pm}_{i,m}\ ,\ a_{i,n}\ ,\ h_i\ ,\ k\quad (1\le i\le M+N-1,\ m\in\mathbb{Z},\ n\in\mathbb{Z}_{\neq 0})$$

Generating functions

$$X^{\pm}_{i}(z)=\sum_{m\in\mathbb{Z}} X^{\pm}_{i,m}\, z^{-m-1}$$

$$\psi^{\pm}_{i}(q^{\frac{k}{2}}z)=q^{\pm h_i}\exp\left(\pm(q-q^{-1})\sum_{m>0} a_{i,m}\, z^{\mp m}\right)$$

Superalgebra $U_q(\widehat{sl}(M|N))$

$[x, y] = xy - yx$

$\{x, y\} = xy + yx$

Relations

- $[a_{i m}, a_{j n}] = \dfrac{[A_{ij} m]_q [k m]_q}{m} \delta_{m+n,0}$

- $[a_{i m}, X_j^{\pm}(z)] = \pm \dfrac{[A_{ij} m]_q}{m} q^{\mp \frac{k}{2}|m|} z^m X_j^{\pm}(z)$

- $\{X_M^+(z_1), X_M^-(z_2)\} = \dfrac{1}{(q-q^{-1}) z_1 z_2} \times$
$$\times \left(\delta\left(q^k \frac{z_2}{z_1}\right) \psi_M^+(q^{\frac{k}{2}} z_2) - \delta\left(q^{-k} \frac{z_2}{z_1}\right) \psi_M^-(q^{-\frac{k}{2}} z_2) \right)$$

- $[X_i^+(z_1), X_j^-(z_2)] = \dfrac{\delta_{ij}}{(q-q^{-1}) z_1 z_2} \times$ $\quad (i, j \neq M)$
$$\times \left(\delta\left(q^k \frac{z_2}{z_1}\right) \psi_i^+(q^{\frac{k}{2}} z_2) - \delta\left(q^{-k} \frac{z_2}{z_1}\right) \psi_i^-(q^{-\frac{k}{2}} z_2) \right)$$

Superalgebra $U_q(\widehat{sl}(M|N))$

Relations

- $(z_1 - q^{\pm A_{ij}} z_2) X_i^{\pm}(z_1) X_j^{\pm}(z_2) = (q^{\pm A_{ij}} z_1 - z_2) X_j^{\pm}(z_2) X_i^{\pm}(z_1)$ $\quad (|A_{ij}| \neq 0)$

- $\{ X_M^{\pm}(z_1), X_j^{\pm}(z_2) \} = 0$

- $[X_i^{\pm}(z_1), X_j^{\pm}(z_2)] = 0$ $\quad (|A_{ij}| = 0, \ i \neq M)$

- $\{ X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_j^{\pm}(z\ ) - (q + q^{-1}) X_i^{\pm}(z_1) X_j^{\pm}(z\cdot) X_i^{\pm}(z_2)$
  $+ X_j^{\pm}(z) X_i^{\pm}(z_1) X_i^{\pm}(z_2) \} + \{ z_1 \Leftrightarrow z_2 \} = 0$
  $(|A_{ij}| = 1, \ i \neq M)$

Superalgebra $U_q(\widehat{sl}(M|N))$

Relations

$$\cdot \Big\{ X_M^{\pm}(z_1) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) - q^{-1} X_M^{\pm}(z_1) X_{M+1}^{\pm}(w_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2)$$
$$- q X_M^{\pm}(z_1) X_M^{\pm}(z_2) X_M^{\pm}(w_2) X_{M+1}^{\pm}(w_1) + X_M^{\pm}(z_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_{M+1}^{\pm}(w_1)$$
$$+ X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_1) - q^{-1} X_{M+1}^{\pm}(w_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_M^{\pm}(z_1)$$
$$- q X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_1) + X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_1) \Big\}$$
$$+ \{ z_1 \leftrightarrow z_2 \} = 0$$

$$\cdot \ [\, k, U_q(\widehat{sl}(M|N)) \,] = 0$$

Level-k Bosonization

$U_q(\widehat{sl}(M|N))$

Boson $a_m^i \ (1 \leq i \leq M+N-1)$ $b_m^{ij}, c_m^{ij} \ (1 \leq i < j \leq M+N)$

$$\cdot \ [a_m^i, a_n^j] = \frac{[(k+M-N)m]_q [A_{ij} m]_q}{m} \delta_{m+n,0}$$

$$\cdot \ [b_m^{ij}, b_n^{i'j'}] = -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{ii'} \delta_{jj'} \delta_{m+n,0}$$

$$\cdot \ [c_m^{ij}, c_n^{i'j'}] = \nu_i \nu_j \frac{[m]_q^2}{m} \delta_{ii'} \delta_{jj'} \delta_{m+n,0}$$

$$\nu_1 = \nu_2 = \cdots = \nu_M = +, \quad \nu_{M+1} = \nu_{M+2} = \cdots = \nu_{M+N} = -$$

Level-k Bosonization

$U_q(\widehat{sl}(M|N))$

Zero-Mode $Q_b^{\bar{i}\bar{j}}$, $Q_c^{\bar{i}\bar{j}}$

- $[b_0^{\bar{i}\bar{j}}, Q_b^{\bar{i}'\bar{j}'}] = -\nu_{\bar{i}}\nu_{\bar{j}}\delta_{\bar{i}\bar{i}'}\delta_{\bar{j}\bar{j}'}$
- $[c_0^{\bar{i}\bar{j}}, Q_c^{\bar{i}'\bar{j}'}] = \nu_{\bar{i}}\nu_{\bar{j}}\delta_{\bar{i}\bar{i}'}\delta_{\bar{j}\bar{j}'}$

Cocycle Condition

- $[Q_b^{\bar{i}\bar{j}}, Q_b^{\bar{i}'\bar{j}'}] = \log(\pi\sqrt{-1})$ $\quad (\nu_{\bar{i}}\nu_{\bar{j}} = \nu_{\bar{i}'}\nu_{\bar{j}'} = -)$

Level-k Bosonization

- $b^{\bar{i}\bar{j}}(z) = -\sum_{m\neq 0} \frac{b_m^{\bar{i}\bar{j}}}{[m]} z^{-m} + Q_b^{\bar{i}\bar{j}} + b_0^{\bar{i}\bar{j}} \log z$

- $c^{\bar{i}\bar{j}}(z) = -\sum_{m\neq 0} \frac{c_m^{\bar{i}\bar{j}}}{[m]} z^{-m} + Q_c^{\bar{i}\bar{j}} + c_0^{\bar{i}\bar{j}} \log z$

- $b_\pm^{\bar{i}\bar{j}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} b_m^{\bar{i}\bar{j}} z^{-m} \pm b_0^{\bar{i}\bar{j}} \log q$

- $a_\pm^{\bar{i}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} a_m^{\bar{i}} z^{-m} \pm a_0^{\bar{i}} \log q$

Level − k Bosonization $\quad U_q(\widehat{sl}(M|N))$

$$X_i^+(z) = \frac{1}{(q-q^{-1})z}\Big[\sum_{j=1}^{i} :\exp\big((b+c)^{j,i}(q^{j-1}z)\big)$$

$$\times\Big\{\exp\big(b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j}z)\big)$$

$$-\exp\big(b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z)\big)\Big\} \qquad (1 \le i \le M-1)$$

$$\times\exp\Big(\sum_{l=1}^{j-1}\big(b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^{l}z)\big)\Big):\Big],$$

$$X_M^+(z) = \sum_{j=1}^{M} :\exp\Big((b+c)^{j,M}(q^{j-1}z) + b^{j,M+1}(q^{j-1}z)$$

$$-\sum_{l=1}^{j-1}\big(b_+^{l,M+1}(q^{l}z) + b_+^{l,M}(q^{l}z)\big)\Big):.$$

Level-k Bosonization

$U_q(\widehat{sl}(M|N))$

$$
\begin{aligned}
X_i^{+}(z) = \sum_{j=1}^{M} :\exp\Big(& b_+^{j,i}(q^{j-1}z) - b^{j,i}(q^{j}z) + b^{j,i+1}(q^{j-1}z) \\
& + \sum_{l=1}^{j-1}\big(b_+^{l,i}(q^{l-1}z) - b_+^{l,i+1}(q^{l}z)\big)\Big): \\
+ \frac{1}{(q-q^{-1})z}\Big[\sum_{j=M+1}^{i} :\Big\{ & \exp\big(-b_+^{j,i+1}(q^{2M+1-j}z) - (b+c)^{j,i+1}(q^{2M-j}z)\big) \\
& - \exp\big(-b_-^{j,i+1}(q^{2M+1-j}z) - (b+c)^{j,i+1}(q^{2M+2-j}z)\big)\Big\} \\
& \times \exp\big((b+c)^{j,i}(q^{2M+1-j}z)\big) \\
\times \exp\Big(\sum_{l=1}^{M}\big(b_+^{l,i}(q^{l-1}z) - b_+^{l,i+1}(q^{l}z)\big) & + \sum_{l=M+1}^{j-1}\big(b_+^{l,i}(q^{2M-l}z) - b_+^{l,i+1}(q^{2M+1-l}z)\big)\Big):\Big]
\end{aligned}
$$

$(M+1 \le i \le M+N-1)$

$:\exp\left(\pm b^{\bar{i}\bar{j}}(z)\right):$ $\quad (v_i v_j = -1)$

$$\left\{ :e^{b^{\bar{i}\bar{j}}(z_1)}:\, ,\, :e^{-b^{\bar{i}\bar{j}}(z_2)}: \right\} = \frac{1}{z_1}\delta\left(z_2/z_1\right)$$

$:\exp\left(b_{\pm}^{\bar{i}\bar{j}}(z) \pm (b+c)^{\bar{i}\bar{j}}(q^{\mp 1}z) \right):$ $\quad (v_i v_j = +1)$

$$\left[:e^{b_{\pm}^{\bar{i}\bar{j}}(z_1) - (b+c)^{\bar{i}\bar{j}}(q^{\pm 1}z_1)}:\, ,\, :e^{b_{\pm}^{\bar{i}\bar{j}}(z_2) + (b+c)^{\bar{i}\bar{j}}(q^{\mp 1}z_2)}: \right]$$

$$= \mp (q - q^{-1})\, \delta\left(q^{\mp 2}\frac{z_2}{z_1}\right) :e^{b_{\pm}^{\bar{i}\bar{j}}(z_1) + b_{\pm}^{\bar{i}\bar{j}}(z_2)}:$$

Superalgebra $U_q(\widehat{sl}(M|N))$

$[x, y] = xy - yx$

$\{x, y\} = xy + yx$

Relations

- $[a_{im}, a_{jn}] = \dfrac{[A_{ij}m]_q [km]_q}{m} \delta_{m+n,0}$

- $[a_{im}, X_j^{\pm}(z)] = \pm \dfrac{[A_{ij}m]_q}{m} q^{\mp\frac{k}{2}|m|} z^m X_j^{\pm}(z)$

- $\{X_M^{+}(z_1), X_M^{-}(z_2)\} = \dfrac{1}{(q-q^{-1})z_1 z_2} \times \left(\delta\left(q^{k}\dfrac{z_2}{z_1}\right) \psi_M^{+}(q^{\frac{k}{2}} z_2) - \delta\left(q^{-k}\dfrac{z_2}{z_1}\right) \psi_M^{-}(q^{-\frac{k}{2}} z_2) \right)$

- $[X_i^{+}(z_1), X_j^{-}(z_2)] = \dfrac{\delta_{ij}}{(q-q^{-1})z_1 z_2} \times \left(\delta\left(q^{k}\dfrac{z_2}{z_1}\right) \psi_i^{+}(q^{\frac{k}{2}} z_2) - \delta\left(q^{-k}\dfrac{z_2}{z_1}\right) \psi_i^{-}(q^{-\frac{k}{2}} z_2) \right)$

Level-k Bosonization $\quad g = M - N$ dual Coxter number

$$X_{i}^{-}(z) = \frac{1}{(q-q^{-1})z}\Big[\sum_{j=1}^{i-1} :\Big\{ \exp\Big(-b_{-}^{ji}(q^{-k-j}z) - (b+c)^{ji}(q^{-k-j+1}z)\Big)$$

$$- \exp\Big(-b_{+}^{ji}(q^{-k-j}z) - (b+c)^{ji}(q^{-k-j-1}z)\Big)$$

$$\times \exp\Big(a_{-}^{i}(q^{-\frac{k+g}{2}}z) + \sum_{l=j+1}^{i}\big(b_{-}^{l\,i+1}(q^{-k-l+1}z) - b_{-}^{l\,i}(q^{-k-l}z)\big)\Big)$$

$$\times \exp\Big(\sum_{l=i+1}^{M}\big(b_{-}^{il}(q^{-k-l}z) - b_{-}^{i+1\,l}(q^{-k-l+1}z)\big) + \sum_{l=M+1}^{M+N}\big(b_{-}^{il}(q^{-k-2M-1+l}z) - b_{-}^{i+1\,l}(q^{-k-2M+l}z)\big)\Big):$$

$$+ \frac{1}{(q-q^{-1})z}\Big\{ :\exp\Big(b_{-}^{i\,i+1}(q^{-k-i+1}z) + (b+c)^{i\,i+1}(q^{-k-i+2}z) + a_{-}^{i}(q^{-\frac{k+g}{2}}z)$$

$$+ \sum_{l=i+2}^{M}\big(b_{-}^{il}(q^{-k-l}z) - b_{-}^{i+1\,l}(q^{-k-l+1}z)\big) + \sum_{l=M+1}^{M+N}\big(b_{-}^{il}(q^{-k-2M-1+l}z) - b_{-}^{i+1\,l}(q^{-k-2M+l}z)\big)\Big):$$

$$- :\exp\Big(b_{+}^{i\,i+1}(q^{k+i-1}z) + (b+c)^{i\,i+1}(q^{k+i-2}z) + a_{+}^{i}(q^{\frac{k+g}{2}}z)$$

$$+ \sum_{l=i+2}^{M}\big(b_{+}^{il}(q^{k+l}z) - b_{+}^{i+1\,l}(q^{k+l-1}z)\big) + \sum_{l=M+1}^{M+N}\big(b_{+}^{il}(q^{k+2M+1-l}z) - b_{+}^{i+1\,l}(q^{k+2M-l}z)\big)\Big):\Big\}$$

$$+ \cdots\cdots \qquad (1 \le i \le M-1)$$

$$\cdots + \frac{1}{(q-q^{-1})z}\Big[\sum_{\bar{j}=\bar{i}+2}^{M} : \Big\{ \exp\Big(b_+^{\bar{i}+1\,\bar{j}}(q^{k+\bar{j}-1}z) - (b+c)^{\bar{i}+1\,\bar{j}}(q^{k+\bar{j}}z)\Big)$$

$$- \exp\Big(b_-^{\bar{i}+1\,\bar{j}}(q^{k+\bar{j}-1}z) - (b+c)^{\bar{i}+1\,\bar{j}}(q^{k+\bar{j}-2}z)\Big)\Big\}$$

$$\times \exp\Big(a_+^{\bar{i}}(q^{\frac{k+g}{2}}z) + (b+c)^{\bar{i}\bar{j}}(q^{k+\bar{j}-1}z)$$

$$+ \sum_{l=\bar{j}}^{M}\big(b_+^{\bar{i}l}(q^{k+l}z) - b_+^{\bar{i}+1\,l}(q^{k+l-1}z)\big) + \sum_{l=M+1}^{M+N}\big(b_+^{\bar{i}l}(q^{k+2M+1-l}z) - b_+^{\bar{i}+1\,l}(q^{k+2M-l}z)\big)\Big):\Big]$$

$$+ \sum_{\bar{j}=M+1}^{M+N} : \exp\Big(-b^{\bar{i}\bar{j}}(q^{k+2M-\bar{j}}z) - b_+^{\bar{i}+1\,\bar{j}}(q^{k+2M-\bar{j}}z) + b^{\bar{i}+1\,\bar{j}}(q^{k+2M+1-\bar{j}}z)$$

$$+ a_+^{\bar{i}}(qz) + \sum_{l=\bar{j}+1}^{M+N}\big(b_+^{\bar{i}l}(q^{k+2M+1-l}z) - b_+^{\bar{i}+1\,l}(q^{k+2M-l}z)\big)\Big):$$

$$(1 \le \bar{i} \le M-1)$$

$U_q(\widehat{sl}(M|N))$

Level $-k$ Bosonization

$g = M - N$

$$X_M^-(z) = \frac{1}{(q-q^{-1})z}\Big[\sum_{j=1}^{M-1} :\Big\{\exp\Big(-b_-^{jM}(q^{-k-j}z) - (b+c)^{jM}(q^{-k-j+1}z)\Big) - \exp\Big(-b_+^{jM}(q^{-k-j}z) - (b+c)^{jM}(q^{-k-j-1}z)\Big)\Big\}\times$$

$$\times \exp\Big(a_-^M(q^{-\frac{k+g}{2}}z) + b_-^{j\,M+1}(q^{-k-j}z) - b^{j,M+1}(q^{-k-j+1}z)$$

$$-\sum_{l=j+1}^{M-1}\big(b_-^{lM}(q^{-k-l}z) + b_-^{l\,M+1}(q^{-k-l}z)\big) + \sum_{l=M+2}^{M+N}\big(b_-^{Ml}(q^{-k-2M-1+l}z) - b_-^{M+1\,l}(q^{-k-2M-1+l}z)\big)\Big):\Big]$$

$$+\frac{1}{(q-q^{-1})z}\Big\{:\exp\Big(a_-^M(q^{-\frac{k+g}{2}}z) - b^{M\,M+1}(q^{-k-M+1}z) + \sum_{l=M+2}^{M+N}\big(b_-^{Ml}(q^{-k-2M-1+l}z) + b_-^{M+1\,l}(q^{-k-2M-1+l}z)\big)\Big):$$

$$- :\exp\Big(a_+^M(q^{\frac{k+g}{2}}z) - b^{M\,M+1}(q^{k+M-1}z) + \sum_{l=M+2}^{M+N}\big(b_+^{Ml}(q^{k+2M+1-l}z) + b_+^{M+1\,l}(q^{k+2M+1-l}z)\big)\Big):\Big\}$$

$$+\frac{1}{(q-q^{-1})z}\Big[\sum_{j=M+2}^{M+N} :\Big\{\exp\Big(-b_-^{M+1\,j}(q^{k+2M-j}z) - (b+c)^{M+1\,j}(q^{k+2M-j+1}z)\Big)$$

$$- \exp\Big(-b_+^{M+1\,j}(q^{k+2M-j}z) - (b+c)^{M+1\,j}(q^{k+2M-j-1}z)\Big)\Big\}\times\exp\Big(-b^{Mj}(q^{k+2M-j}z)$$

$$+a_+^M(q^{\frac{k+g}{2}}z) + b_+^{M+1\,j}(q^{k+2M-j}z) + \sum_{l=j+1}^{M+N}\big(b_+^{Ml}(q^{k+2M+l-1}z) + b_+^{M+1\,l}(q^{k+2M+l-1}z)\big)\Big):\Big].$$

Level-k Bosonization $\qquad g = M - N$

$$X_i^-(z) = \sum_{j=1}^{M} :\exp\Big(b_-^{j\,i+1}(q^{-k-j} z) - b^{j\,i+1}(q^{-k-j+1} z) + b^{j\,i}(q^{-k-j} z) + a_-^{i}(q^{-\frac{k+g}{2}} z)$$

$$+ \sum_{l=j+1}^{M} \big(b_-^{l\,i}(q^{-k-l+1} z) - b_-^{l\,i+1}(q^{-k-l} z) \big) + \sum_{l=M+1}^{i} \big(b_-^{l\,i}(q^{-k-2M+l} z) - b_-^{l\,i+1}(q^{-k-2M-1+l} z) \big)$$

$$+ \sum_{l=i+1}^{M+N} \big(b_-^{i+1\,l}(q^{-k-2M-1+l} z) - b_-^{i\,l}(q^{-k-2M+l} z) \big) \Big):$$

$$+ \frac{1}{(q-q^{-1})z} \Big[\sum_{j=M+1}^{i-1} :\Big\{ \exp\big(b_-^{j\,i}(q^{-k-2M+j} z) - (b+c)^{j\,i}(q^{-k-2M+j+1} z) \big)$$

$$- \exp\big(b_+^{j\,i}(q^{-k-2M+j} z) - (b+c)^{j\,i}(q^{-k-2M+j-1} z) \big) \Big\}$$

$$\times \exp\Big(a_-^{i}(q^{-\frac{k+g}{2}} z) + (b+c)^{j\,i+1}(q^{-k-2M+1+j} z) +$$

$$+ \sum_{l=j+1}^{i} \big(b_-^{l\,i}(q^{-k-2M+l} z) - b_-^{l\,i+1}(q^{-k-2M+l-1} z) \big)$$

$$+ \sum_{l=i+1}^{M+N} \big(b_-^{i+1\,l}(q^{-k-2M-1+l} z) - b_-^{i\,l}(q^{-k-2M+l} z) \big) \Big): \Big] + \cdots$$

$$(M+1 \le i \le M+N-1)$$

''''

$$
+\frac{1}{(q-q^{-1})z}:\Big\{\exp\Big(-b_-^{i,i+1}(q^{-k-2M+1+i}z)+(b+c)^{i,i+1}(q^{-k-2M+i}z)+a_-^{i}(q^{-\frac{k+g}{2}}z)
$$

$$
+\sum_{l=i+2}^{M+N}\big(b_-^{i+1,l}(q^{-k-2M-1+l}z)-b_-^{i,l}(q^{-k-2M+l}z)\big)\Big)
$$

$$
-\exp\Big(-b_+^{i,i+1}(q^{k+2M-1-i}z)+(b+c)^{i,i+1}(q^{k+2M-i}z)+a_+^{i}(q^{\frac{k+g}{2}}z)
$$

$$
+\sum_{l=i+2}^{M+N}\big(b_+^{i+1,l}(q^{k+2M+1-l}z)-b_+^{i,l}(q^{k+2M-l}z)\big)\Big)\Big\}:
$$

$$
+\frac{1}{(q-q^{-1})z}\Big[\sum_{j=i+2}^{M+N}:\Big\{\exp\Big(-b_-^{i+1,j}(q^{k+2M+1-j}z)-(b+c)^{i+1,j}(q^{k+2M-j+2}z)\Big)
$$

$$
-\exp\Big(-b_+^{i+1,j}(q^{k+2M+1-j}z)-(b+c)^{i+1,j}(q^{k+2M-j}z)\Big)\Big\}
$$

$$
\times\exp\Big(a_+^{i}(q^{\frac{k+g}{2}}z)+b_+^{i+1,j}(q^{k+2M+1-j}z)
$$

$$
-b_+^{i,j}(q^{k+2M-j}z)+(b+c)^{i,j}(q^{k+2M-j}z)+\sum_{l=j+1}^{M+N}\big(b_+^{i+1,l}(q^{k+2M+1-l}z)-b_+^{i,l}(q^{k+2M-l}z)\big)\Big):\Big]
$$

$$
(M+1\le i\le M+N-1)
$$

Level-k Bosonization $\quad U_q(\widehat{sl}(M|N))$

$$
\begin{cases}
X^{+,i}(z) = \sum_{j=1}^{i} \frac{1}{(q-q^{-1})z} \left(E^{+}_{ij}(z) - E^{-}_{ij}(z) \right) & (1 \le i \le M-1) \\
X^{+,M}(z) = \sum_{j=1}^{M} E_{Mj}(z) & (i = M) \\
X^{+,i}(z) = \sum_{j=1}^{M} E_{ij}(z) + \sum_{j=M+1}^{i} \frac{1}{(q-q^{-1})z} \left(E^{+}_{ij}(z) - E^{-}_{ij}(z) \right) & (M+1 \le i \le M+N-1)
\end{cases}
$$

$$
\begin{cases}
X^{-,i}(z) = \sum_{j=1}^{i} \frac{1}{(q-q^{-1})z} \left(F^{1-}_{ij}(z) - F^{1+}_{ij}(z) \right) & (1 \le i \le M-1) \\
\qquad + \sum_{j=i+2}^{M} \frac{1}{(q-q^{-1})z} \left(F^{2-}_{ij}(z) - F^{2+}_{ij}(z) \right) + \sum_{j=M+1}^{M+N} F^{2}_{ij}(z) & \\
X^{-,M}(z) = \sum_{j=1}^{M} \frac{1}{(q-q^{-1})z} \left(F^{1-}_{M,j}(z) - F^{1+}_{M,j}(z) \right) + \sum_{j=M+2}^{M+N} \frac{1}{(q-q^{-1})z} \left(F^{2-}_{M,j}(z) - F^{2+}_{M,j}(z) \right) & \\
X^{-,i}(z) = \sum_{j=1}^{M} F^{1}_{ij}(z) + \sum_{j=M+1}^{i} \frac{1}{(q-q^{-1})z} \left(F^{1-}_{ij}(z) - F^{1+}_{ij}(z) \right) & \\
\qquad + \sum_{j=i+2}^{M+N} \frac{1}{(q-q^{-1})z} \left(F^{2-}_{ij}(z) - F^{2+}_{ij}(z) \right) & (M+1 \le i \le M+N-1)
\end{cases}
$$

Highest weight representation $V(\lambda)$

$$V(\lambda) \subset \bigoplus \mathcal{F}(p_a, p_b, p_c)$$

$p_b^{ij} = -p_c^{ij} \in \mathbb{Z}$ $\quad (\nu_i \nu_j = +)$

$p_b^{ij} \in \mathbb{Z}$ $\quad (\nu_i \nu_j = -)$

$|\lambda\rangle = |p_a, 0, 0\rangle \in V(\lambda)$ highest weight vector

with $\bar{\lambda} = \sum_{j=1}^{M+N} p_a^j \bar{\Lambda}_j$

$$a_0^i |p_a p_b p_c\rangle = p_a^i |p_a p_b p_c\rangle$$

$$b_0^{ij} |p_a p_b p_c\rangle = p_b^{ij} |p_a p_b p_c\rangle, \quad c_0^{ij} |p_a p_b p_c\rangle = p_c^{ij} |p_a p_b p_c\rangle$$

$$\mathcal{F}(p_a p_b p_c) = \mathbb{C}[a_{-m}^i, b_{-m}^{ij}, c_{-m}^{ij}] |p_a, p_b, p_c\rangle$$

Supporting Argument : $q \mapsto 1$

- Taking the limit $q \mapsto 1$, we get a bosonization of $\widehat{sl}(M|N)$ for level k.
- Because $b_{\pm}^{IJ}(z) \mapsto 0$ $(q \mapsto 1)$, simplification occures.

$$
\begin{aligned}
&:\exp\Big((b+c)^{J,M}(q^{J-1}z) + b^{J,M+1}(q^{J-1}z) \\
&\qquad - \sum_{l=1}^{J-1} \big(b_+^{l,M+1}(q^l z) + b_+^{l,M}(q^l z) \big) \Big): \\
&\longmapsto \; :\exp\Big((b+c)^{J,M}(z) + b^{J,M+1}(q^{J-1}z) \Big):
\end{aligned}
$$

In the limit $q \mapsto 1$

- $X^{+,\bar{i}}(z) = \sum_{\bar{j}=1}^{\bar{i}} :\beta_{\bar{j},\bar{i}+1}(z)\,\gamma_{\bar{j}\bar{i}}(z): \qquad (1 \le \bar{i} \le M-1)$
- $X^{+,M}(z) = \sum_{\bar{j}=1}^{M} :\gamma_{\bar{j},M}(z)\,\psi_{\bar{j},M+1}(z):$
- $X^{+,\bar{i}}(z) = \sum_{\bar{j}=1}^{M} :\psi_{\bar{j},\bar{i}+1}(z)\,\psi^{+}_{\bar{j},\bar{i}}(z): - \sum_{\bar{j}=M+1}^{\bar{i}} :\beta_{\bar{j},\bar{i}+1}(z)\,\gamma_{\bar{j},\bar{i}}(z):$

etc. $\qquad (M+1 \le \bar{i} \le M+N-1)$

$$\beta_{\bar{i}\bar{j}}(z) =: \partial_z\left(e^{-c^{\bar{i}\bar{j}}(z)}\right) e^{-b^{\bar{i}\bar{j}}(z)}$$

$$\gamma_{\bar{i}\bar{j}}(z) =: e^{(b+c)^{\bar{i}\bar{j}}(z)}:$$

$$\psi_{\bar{i}\bar{j}}(z) =: e^{b^{\bar{i}\bar{j}}(z)}: \quad , \quad \psi^{+}_{\bar{i}\bar{j}}(z) =: e^{-b^{\bar{i}\bar{j}}(z)}:$$

quantization is nontrivial.

Supporting Argument from $U_q(sl(M|N))$

$$U_q(\widehat{sl}(M|N)) \supset U_q(sl(M|N))$$

A Realization of $U_q(sl(M|N))$

$$\nu_1 = \nu_2 = \cdots = \nu_M = +, \quad \nu_{M+1} = \nu_{M+2} = \cdots = \nu_{M+N} = -$$

- $x_{ij} = \begin{cases} z_{ij} & (\nu_i \nu_j = +) \\ \theta_{ij} & (\nu_i \nu_j = -) \end{cases}$ $\quad [z_{ij}, z_{kl}] = 0$, $\quad \{\theta_{ij}, \theta_{kl}\} = 0$

- $D_{ij} = x_{ij} \dfrac{\partial}{\partial x_{ij}}$

[Awata, Odake, Shiraishi : LMP (1997)]

Realization $U_q(sl(M|N))$ $(\lambda_i \in \mathbb{C})$

$$E_i = \sum_{j=1}^{i} x_{j\,i} \frac{1}{x_{j\,i+1}} [D_{j\,i+1}]_q \, q^{\sum_{l=1}^{j-1} (\nu_i D_{l\,i} - \nu_{i+1} D_{l\,i+1})}$$

$$F_i = \sum_{j=1}^{i-1} \nu_i \, x_{j\,i+1} \frac{1}{x_{j\,i}} [D_{j\,i}]_q \, q^{-\lambda_i + (\nu_i + \nu_{i+1}) D_{i\,i+1}}$$
$$\times \, q^{\sum_{l=j+1}^{i-1} (\nu_{i+1} D_{l\,i+1} - \nu_i D_{l\,i}) + \sum_{l=i+2}^{M+N} (\nu_i D_{i\,l} - \nu_{i+1} D_{i+1\,l})}$$
$$+ \, x_{i\,i+1} \left[\lambda_i - \nu_i D_{i\,i+1} - \sum_{l=i+2}^{M+N} (\nu_i D_{i\,l} - \nu_{i+1} D_{i+1\,l})\right]_q$$
$$- \sum_{j=i+1}^{M+N} \nu_{i+1} \, x_{i\,j+1} \frac{1}{x_{i+1\,j+1}} [D_{i+1\,j+1}]_q \, q^{\lambda_i + \sum_{l=j+1}^{M+N} (\nu_{i+1} D_{i+1\,l} - \nu_i D_{i\,l})} .$$

Example

$U_q(sl(M|N))$

Replacement

$$F_{N,j} \longmapsto \frac{1}{(q-q^{-1})z} \times$$

$$\times : \exp\Big(-a_-^{N}(z) - b^{j,N+1}(z) - (b+c)^{j,N}(z) + \sum_{l=j+1}^{N-1}\big(b_-^{l,N+1}(z) - b_-^{l,N}(z)\big)\Big)$$

$$\times \Big(\exp\big(-b_+^{j,N}(z) - b^{j,N+1}(z)\big) - \exp\big(-b_-^{j,N}(z) - b^{j,N+1}(z)\big)\Big):.$$

$$\longmapsto \frac{1}{(q-q^{-1})z} \times$$

$$\times : \exp\Big(-a_-^{N}(q^{-\frac{k+N-1}{2}}z) - b^{j,N+1}(q^{-k-j+1}z) - (b+c)^{j,N}(q^{-k-j+1}z) + \sum_{l=j+1}^{N-1}\big(b_-^{l,N+1}(q^{-k-l}z) - b_-^{l,N}(q^{-k-l}z)\big)\Big)$$

$$\times \Big(\exp\big(-b_+^{j,N}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j-1}z)\big) - \exp\big(-b_-^{j,N}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j+1}z)\big)\Big):.$$

$U_q(\widehat{sl}(M|N))$

Replacement

$$q^{D_{ij}} \longmapsto e^{\pm b^{ij}_{\mp}(z)}$$

$$[D_{ij}]_q \longmapsto \begin{cases} \dfrac{e^{\pm b^{ij}_{\mp}(z)} - e^{\pm b^{ij}_{\pm}(z)}}{(q-q^{-1})z} & (v_i v_j = +) \\ 1 & (v_i v_j = -) \end{cases}$$

$$x_{ij} \longmapsto \begin{cases} :e^{(b+c)^{ij}(z)}: & (v_i v_j = +) \\ :e^{-b^{ij}(z)}: \quad \text{or} \quad :e^{-b^{ij}_{\pm}(z) - b^{ij}(z)}: & (v_i v_j = -) \end{cases}$$

$$q^{\lambda_i} \longmapsto e^{\pm a^{i}_{\pm}(z)}$$

$$[\lambda_i]_q \longmapsto \frac{e^{\pm a^{i}_{\mp}(z)} - e^{\pm a^{i}_{\pm}(z)}}{(q-q^{-1})z}$$

- We constructed a bosonization of the superalgebra $U_q(\widehat{sl}(M|N))$ for Level $k \in \mathbb{C}$

$U_q(\widehat{sl}(2|1))$ [Awata, Odake, Shiraishi : LMP (1997)]

$U_q(\widehat{sl}(M|1))$ [Kojima : JMP (2012)]

$U_q(\widehat{sl}(M|N))$ [Kojima : arXiv. 1701.03645 (2017)]
CMP (2017)

Part II : Recent Progress

§2 Screenings of $U_q(\widehat{\mathfrak{sl}}(M|N))$

$$[Q_i, U_q(\widehat{\mathfrak{sl}}(M|N))] = 0$$

Screening

$Q_i \quad (1 \leq i \leq M+N-1)$

- Our bosonization is not irreducible.
- There exists non-trivial (non-constant) operator that commutes with $U_q(\widehat{sl}(M|N))$.

$$[Q_i, U_q(\widehat{sl}(M|N))] = 0$$

Schur's Lemma

Screening Current

$U_q(\widehat{sl}(M|N))$, $g = M - N$.

$$[\, a_{m,i} \,,\, S_j(z) \,] = 0$$

$$[\, X_i^{+}(z_1) \,,\, S_j(z_2) \,] = 0 \,, \quad \{ X_M^{+}(z_1) , S_M(z_2) \} = 0$$

$$[\, X_i^{-}(z_1) \,,\, S_j(z_2) \,] = \frac{\delta_{ij}}{(q - q^{-1}) z_1 z_2} \times$$

$$\times \left(\delta\left(q^{k+g} \frac{z_2}{z_1}\right) - \delta\left(q^{-k-g} \frac{z_2}{z_1}\right) \right) : e^{-\left(\frac{1}{k+g} a^{j}\right)\left(z_1 \middle| -\frac{k+g}{2}\right)} :$$

$$\{ X_M^{-}(z_1) , S_M(z_2) \} = \frac{1}{(q - q^{-1}) z_1 z_2} \times$$

$$\times \left(\delta\left(q^{k+g} \frac{z_2}{z_1}\right) - \delta\left(q^{-k-g} \frac{z_2}{z_1}\right) \right) : e^{-\left(\frac{1}{k+g} a^{M}\right)\left(z_1 \middle| -\frac{k+g}{2}\right)} :$$

Screening Operator

$$Q_{\bar{j}} = \int_0^{s\infty} S_{\bar{j}}(w)\, d_{q^{2(k+g)}} w \qquad (1 \leq \bar{j} \leq M+N-1)$$

$$\Rightarrow \quad [\, Q_{\bar{j}},\ U_q(\widehat{sl}(M|N))\,] = 0$$

$$(\ \{ Q_M,\ X_M^{\pm}(z) \} = 0\ )$$

Jackson Integral

$$\int_0^{s\infty} f(w)\, d_p w = s(1-p) \sum_{m \in \mathbb{Z}} f(sp^m)\, p^m$$

Screening Current

$$\widehat{S}_i(z) = \sum_{\bar{j}=i+1}^{M+N} \frac{1}{(q-q^{-1})z} : \exp\left((b+c)^{i+1,\bar{j}}(q^{-M-N+\bar{j}}z)\right)$$

$$\times \left\{ \exp\left(b_-^{\bar{i}\bar{j}}(q^{-M-N+\bar{j}}z) - (b+c)^{\bar{i}\bar{j}}(q^{-M-N+\bar{j}+1}z) \right) \right.$$

$$\left. - \exp\left(b_+^{\bar{i}\bar{j}}(q^{-M-N+\bar{j}}z) - (b+c)^{\bar{i}\bar{j}}(q^{-M-N+\bar{j}-1}z) \right) \right\} \times$$

$$\times \exp\left(-\sum_{l=\bar{j}+1}^{M+N} \left(b_-^{\bar{i}+1,l}(q^{-M-N+l-1}z) - b_-^{\bar{i},l}(q^{-M-N+l}z) \right) \right) :$$

$$(M+1 \le i \le M+N-1),$$

$$\widehat{S}_M(z) = \sum_{\bar{j}=M+1}^{M+N} : \exp\left((b+c)^{M+1,\bar{j}}(q^{-M-N+\bar{j}}z) \right.$$

$$\left. + b^{M,\bar{j}}(q^{-M-N+\bar{j}}z) - \sum_{l=\bar{j}+1}^{M+N} \left(b_-^{M+1,l}(q^{-M-N+l-2}z) - b_-^{M,l}(q^{-M-N+l-1}z) \right) \right) :$$

Screening Current

$(1 \le i \le M-1)$

$$\widetilde{S}_i(z) = \sum_{j=i+1}^{M} \frac{1}{(q-q^{-1})z} : \exp\left((b+c)^{i+1,j}(q^{M-N-j}z) \right) \times$$

$$\times \left\{ \exp\left(-b_-^{i,j}(q^{M-N-j}z) - (b+c)^{i,j}(q^{M-N-j+1}z) \right) - \exp\left(-b_+^{i,j}(q^{M-N-j}z) - (b+c)^{i,j}(q^{M-N-j-1}z) \right) \right\} \times$$

$$\times \exp\left(\sum_{\ell=j+1}^{M} \left(b_-^{i+1,\ell}(q^{M-N-\ell+1}z) - b_-^{i,\ell}(q^{M-N-\ell}z) \right) + \sum_{\ell=M+1}^{M+N} \left(b_-^{i+1,\ell}(q^{-M-N+\ell}z) - b_-^{i,\ell}(q^{-M-N+\ell-1}z) \right) \right) :$$

$$+ \sum_{j=M+1}^{M+N} : \exp\left(b^{i,j}(q^{-M-N+j}z) + b_+^{i+1,j}(q^{-M-N+j}z) - b^{i+1,j}(q^{-M-N+j+1}z) + \sum_{\ell=j+1}^{M+N} \left(b_-^{i+1,\ell}(q^{-M-N+\ell}z) - b_-^{i,\ell}(q^{-M-N+\ell-1}z) \right) \right) :$$

Screening Operator

Level $k \neq -g = -M+N$

$$Q_{\bar{i}} = \int_0^{s\infty} S_{\bar{i}}(w)\, d_{q^{2(k+g)}} w \qquad \text{Jackson integral}$$

$$S_{\bar{i}}(w) = :\exp\left(-\left(\frac{1}{k+g} a^{\bar{i}}\right)\left(w \middle| \frac{k+g}{2}\right)\right)\widetilde{S}_{\bar{i}}(w): \qquad (1 \leq \bar{i} \leq M+N-1)$$

$$[\, Q_{\bar{i}},\ U_q(\widehat{sl}(M|N))\,] = 0$$

$$\left(\frac{1}{\beta} a^{\bar{i}}\right)(z|\alpha) = -\sum_{m \neq 0} \frac{a_m^{\bar{i}}}{[\beta m]_q [m]_q} q^{-\alpha|m|} z^{-m} + \frac{1}{\beta}\left(Q_a^{\bar{i}} + a_0^{\bar{i}} \log z\right)$$

ξ-η system

$U_q(\widehat{sl}(M|N))$

$$\xi_0 = \prod_{\substack{1 \le i < j \le M+N \\ \nu_i \nu_j = +}} \xi_0^{ij}, \quad \xi_0^{ij} = \frac{1}{2\pi i} \oint : e^{c^{ij}(w)} : dw$$

$$\eta_0 = \prod_{\substack{1 \le i < j \le M+N \\ \nu_i \nu_j = +}} \eta_0^{ij}, \quad \eta_0^{ij} = \frac{1}{2\pi i} \oint : e^{-c^{ij}(w)} : \frac{dw}{w}$$

$$[\eta_0 \xi_0, U_q(\widehat{sl}(M|N))] = 0$$

$$[\eta_0 \xi_0, Q_i] = 0 \quad (1 \le i \le M+N-1)$$

Summary of §2

- We constructed the scheening operators Q_i $(1 \leq i \leq M+N-1)$ and $\eta_0 \xi_0$ such that
  - $[Q_i, U_q(\widehat{sl}(M|N))] = 0$ $\{Q_M, X^{\pm}_M(z)\} = 0$
  - $[\eta_0 \xi_0, U_q(\widehat{sl}(M|N))] = 0$
  - $[Q_i, \eta_0 \xi_0] = 0$

  for Level $k \neq -g$ $(g = M-N)$

Applications of Screening Operators

Examples

① **Irreducible representation** is constructed by Felder complex based on screenings.

② **Background charge** of the vertex operator is balanced by screenings.

Part II Recent Progress

§3 Vertex Operator of $U_q(\widehat{sl}(M|N))$

Vertex Operator

$$\Phi_{V(\mu)}^{V(\nu)\,V_\lambda}(z) : V(\mu) \longrightarrow V(\nu) \otimes V_{\lambda,z}$$

$V(\mu), V(\nu)$: highest weight module

V_λ : typical module

$V_{\lambda,z}$: evaluation module of V_λ

$\{ v_{i_1 i_2 \cdots i_n} \}$ basis of V_λ

$$\Phi_{V(\mu)}^{V(\nu)\,V_\lambda}(z) = \sum_{i_1 i_2 \cdots i_n} \Phi_{V(\mu)\; i_1 i_2 \cdots i_n}^{V(\nu)\,V_\lambda}(z) \otimes v_{i_1 i_2 \cdots i_n}$$

Conjecture

- $$\Phi^{V(\nu)\,V_x}_{V(\mu)\,\bar{i}_1\bar{i}_2\cdots\bar{i}_n}(z) = \eta_0\xi_0 : \prod_{\bar{j}=1}^{M+N-1} Q_{\bar{j}}^{n_{\bar{j}}}\, \phi^{\bar{\lambda}}_{\bar{i}_1\bar{i}_2\cdots\bar{i}_n}(q^{k+g} z) : \eta_0\xi_0$$

- $$\phi^{\bar{\lambda}}_{\bar{i}_1\bar{i}_2\cdots\bar{i}_n}(z) = \left[\phi^{\bar{\lambda}}_{\bar{i}_1\bar{i}_2\cdots\bar{i}_{n-1}}(z), \oint \frac{dw}{2\pi i} X_{\bar{i}_n}(w) \right]_{q^x}$$

$$\left(x = \left(\bar{\lambda} - \sum_{s=1}^{n-1} \bar{\alpha}_{\bar{i}_s} \,\middle|\, \bar{\alpha}_{\bar{i}_n} \right) \right)$$

- $$\phi^{\bar{\lambda}}(z) = : \exp\left(\sum_{\bar{i},\bar{j}=1}^{M+N-1} \left(\frac{l_{\bar{i}}}{k+g} \cdot \frac{\alpha_{\bar{i}\bar{j}}}{g} \cdot \frac{B_{\bar{i}\bar{j}}}{1} \alpha^{\bar{j}} \right) \left(z \,\middle|\, -\frac{k+g}{2} \right) \right) :$$

$$\left(\bar{\lambda} = \sum_{\bar{i}=1}^{M+N-1} l_{\bar{i}} \bar{\Lambda}_{\bar{i}} \right)$$

We checked intertwing property for $M=2,3,4$ and $N=1$.

- $$\left(\begin{matrix}\gamma_1 & \gamma_2 & \cdots & \gamma_s \\ \beta_1 & \beta_2 & \cdots & \beta_s\end{matrix}\; a^{\bar{i}}\right)(z|\alpha)$$

$$= -\sum_{m\neq 0} \frac{[\gamma_1 m]_q \cdots [\gamma_s m]_q}{[\beta_1 m]_q \cdots [\beta_s m]_q} \frac{a_m^{\bar{i}}}{[m]_q} q^{-\alpha|m|} z^{-m}$$

$$+ \frac{\gamma_1 \cdots \gamma_s}{\beta_1 \cdots \beta_s} \left(Q_a^{\bar{i}} + a_0^{\bar{i}} \log z \right)$$

- $$\alpha_{\bar{i}\bar{j}} = \begin{cases} \min(\bar{i},\bar{j}) & (\min(\bar{i},\bar{j}) \leq M+1) \\ 2(M+1) - \min(\bar{i},\bar{j}) & (\min(\bar{i},\bar{j}) \geq M+2) \end{cases}$$

$$\beta_{\bar{i}\bar{j}} = \begin{cases} M-N-\max(\bar{i},\bar{j}) & (\max(\bar{i},\bar{j}) \leq M+1) \\ -M-N-2+\max(\bar{i},\bar{j}) & (\max(\bar{i},\bar{j}) \geq M+2) \end{cases}$$

Summary

- We gave a bosonization of the superalgebra $U_q(\widehat{sl}(M|N))$ for level $k \in \mathbb{C}$.
- We gave the screening operators that commute with $U_q(\widehat{sl}(M|N))$ for $k \neq -M+N$.
- We propose a bosonization of the vertex operator for $k \neq -M+N$

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