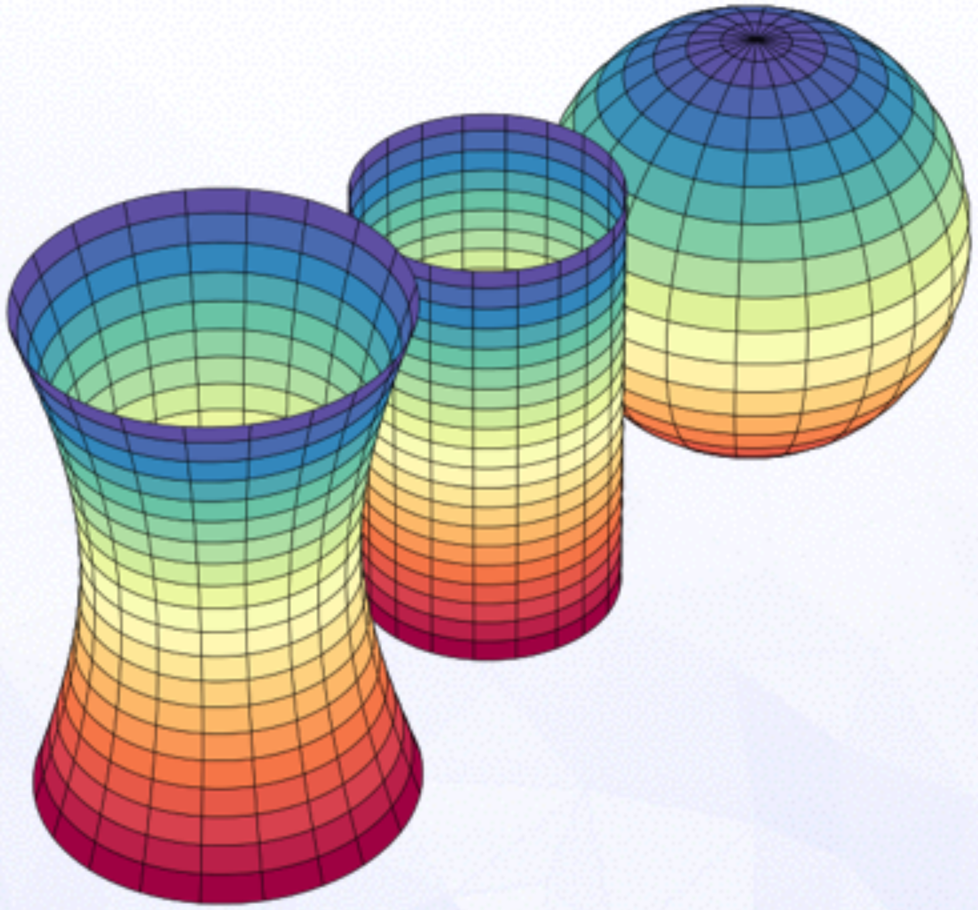

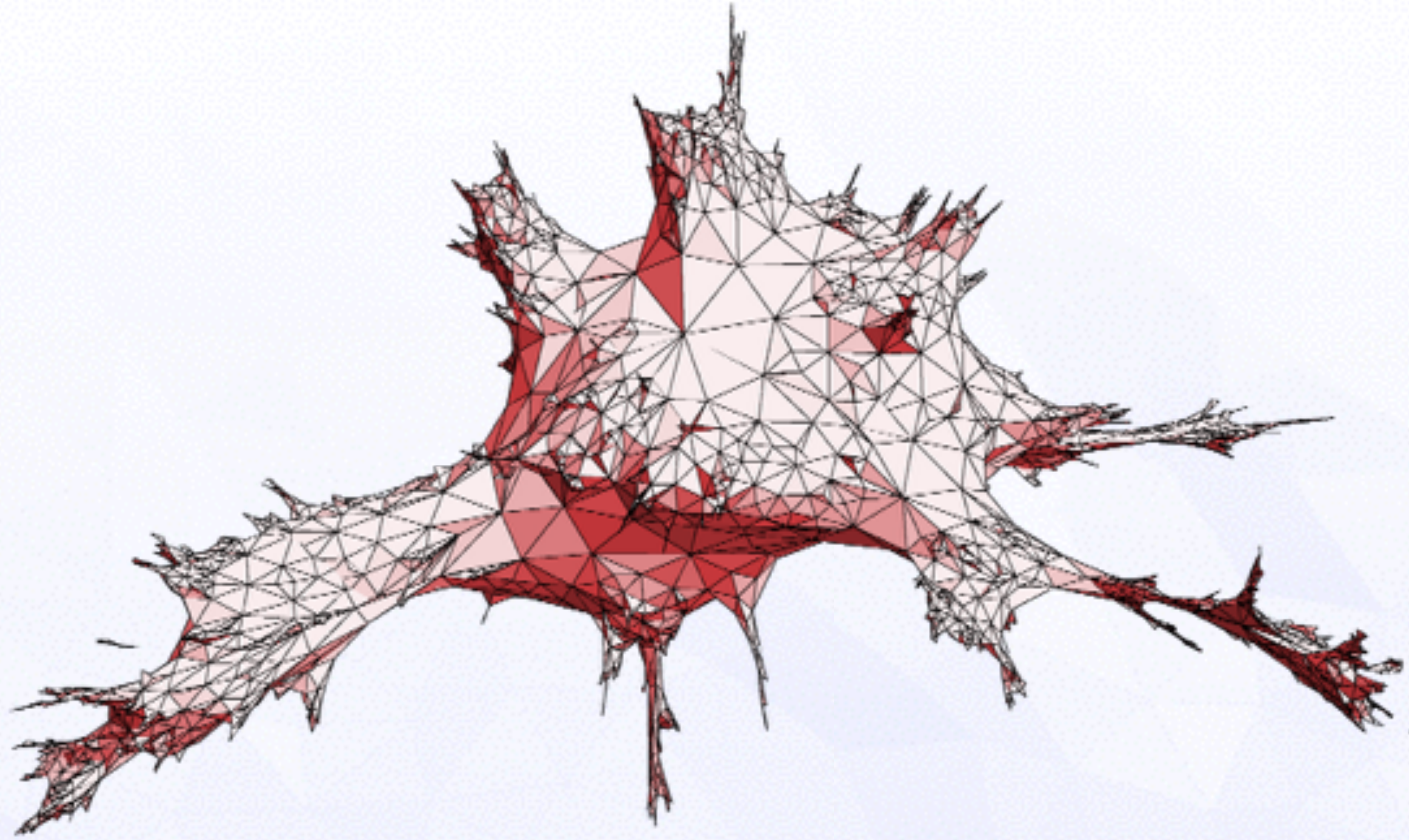

Quantum Ricci Curvature

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M∩Φ, Belgrade **20 Sep 2017**

Exploring spacetime at the Planck scale

Our main goal is to construct a theory of *Quantum Gravity*, a fundamental quantum theory underlying General Relativity.

My talk today is about the new concept of *quantum curvature*, a quasilocal observable in a nonperturbative, Planckian regime, where spacetime is no longer described by a smooth metric $g_{\mu\nu}(x)$.

I will explain the idea in the continuum and then implement it on various piecewise linear (PL) spaces, including the equilateral configurations of *(Causal) Dynamical Triangulations*. CDT is a candidate theory of quantum gravity, based on a non-perturbative path integral, where our "***quantum Ricci curvature***" can be calculated in a straightforward way. However, the concept is applicable much more widely, and you need not know about CDT.

(joint work with Nilas Klitgaard, to be published)

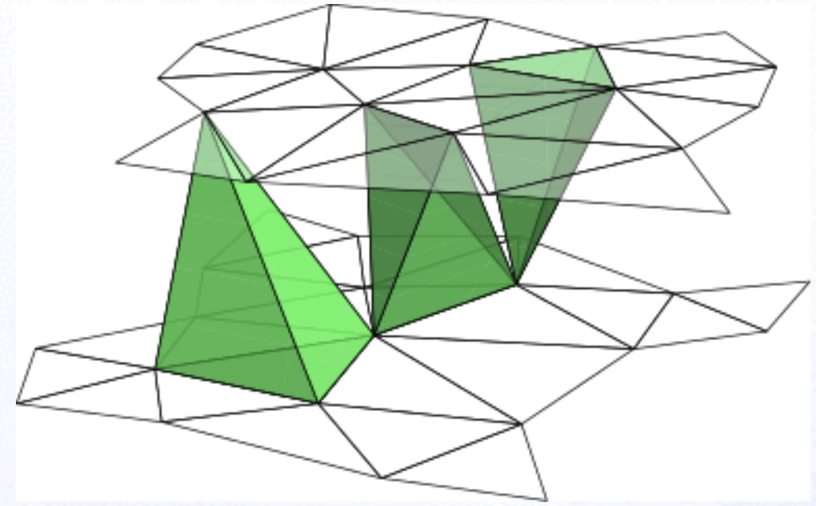
The case for quantum observables

Even assuming we had resolved all "technical difficulties" in specific quantum gravity theories, we still face the key issue of having to construct meaningful ***observables*** to quantify the physical quantum properties of gravity and spacetime (or whatever remains of them) at the Planck scale. These are prior to the development of any true QG phenomenology and should

1. be purely geometrical (coordinate-invariant, background-indept.),
2. have finite, nonzero expectation values in the ensemble,
3. be measurable reliably in the window accessible to quantitative evaluation (simulation or other numerical methods),
4. have a (semi-)classical limit.

We have such observables, e.g. in CDT, but they are rather coarse (dynamical dimensions, volume profiles) and should be complemented by quantities carrying more local geometric information.

The case for CDT

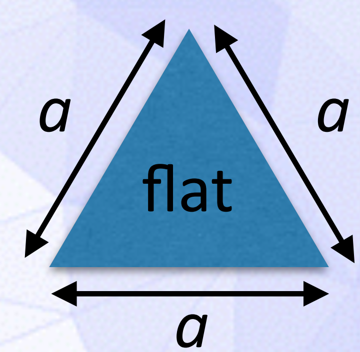

piece of a causal triangulation in 3D

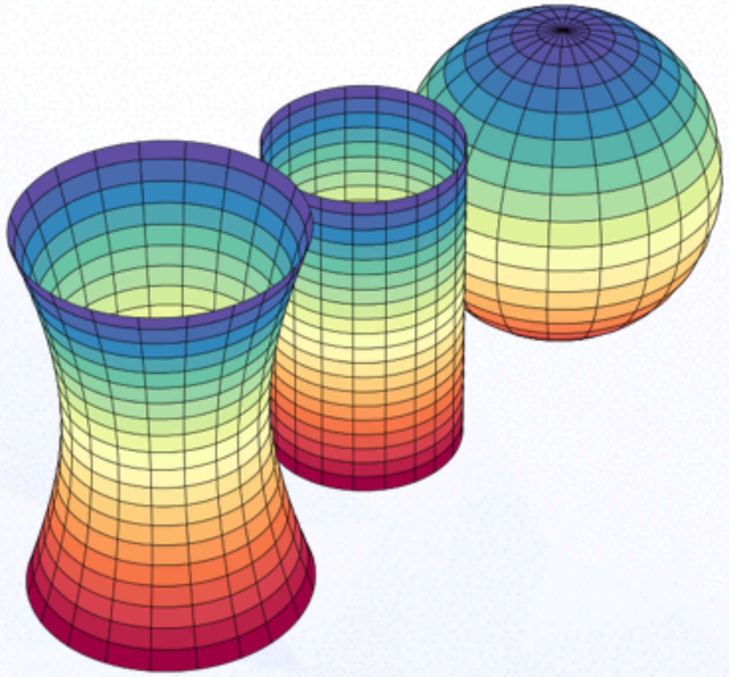
• CDT quantum gravity is a perfect setting to study such nonperturbative quantum observables. Its regularized path integral (“sum over spacetimes”) is defined purely geometrically in terms of simplicial manifolds that are gluings of identical D-dimensional flat simplices (➔ “Random Geometry”). Observables are evaluated quantitatively by Monte Carlo simulation.

• Since lengths and volumes come in discrete units, measuring is often reduced to simple counting. (C)DT geometries are of “Regge type”, i.e. continuous, but with curvature singularities.

• This looks like a conservative setting, but allows for nonclassical behaviour and noncanonical scaling. In fact, we have learned that such behaviour is generic and often prevents the existence of a classical limit when the UV-cutoff *a* is removed.

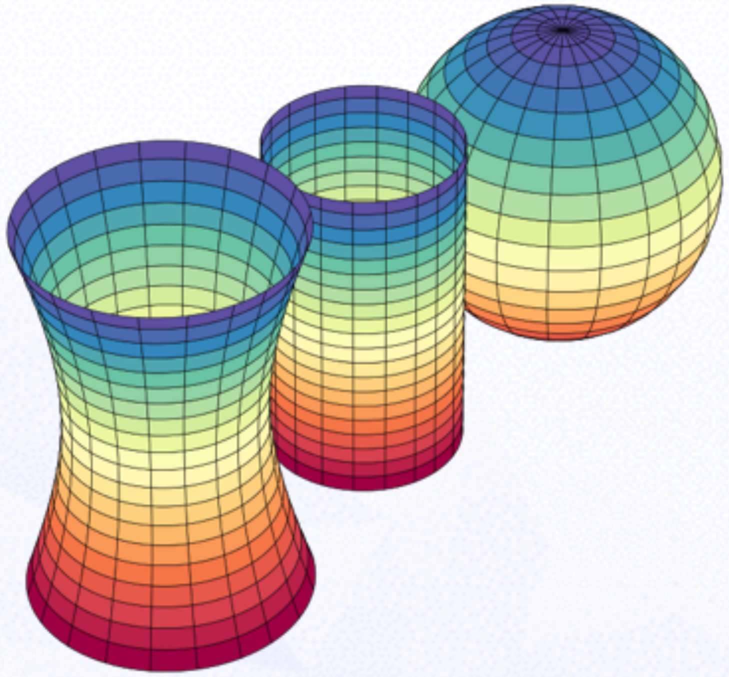
building block in 2D:





The case for curvature

- Curvature is a crucial concept in describing classical spacetime geometry. It is a complex, derived object. Computation of the Riemann tensor $R^{\kappa}{}_{\lambda\mu\nu}[g,\partial g,\partial^2 g;x)$ requires a smooth metric g.
- Finding a meaningful notion of "quantum curvature", applicable in a more general context, has so far received little attention. (Note that definitions in terms of deficit angles are of limited usefulness.)
- Is there a classical characterization of curvature that can be used to obtain a coarse-grained, robust and computable notion of quantum curvature in nonperturbative quantum gravity?



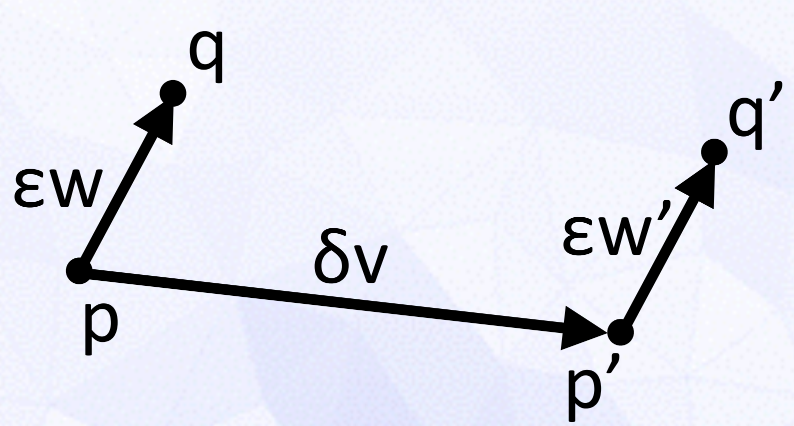
The case for curvature

- Curvature is a crucial concept in describing classical spacetime geometry. It is a complex, derived object. Computation of the Riemann tensor $R^{\kappa}{}_{\lambda\mu\nu}[g,\partial g,\partial^2 g;x)$ requires a smooth metric g.
- Finding a meaningful notion of "quantum curvature", applicable in a more general context, has so far received little attention. (Note that definitions in terms of deficit angles are of limited usefulness.)
- Is there a classical characterization of curvature that can be used to obtain a coarse-grained, robust and computable notion of quantum curvature in nonperturbative quantum gravity? **YES.**
- N.B.: we are working in a Riemannian context ("after Wick rotation")

The key idea (classical):

On a D-dimensional manifold *$(M,g_{\mu\nu})$*, **compare the distance of two small *(D-1)*-spheres with the distance *δ* of their centres.**

Step 1:



v, w unit vectors at *p, $v \perp w$*

$\varepsilon, \delta > 0$

w' is the parallel transport of *w* along *v*

$p' = exp_p(\delta v)$, $q = exp_p(\varepsilon w)$,

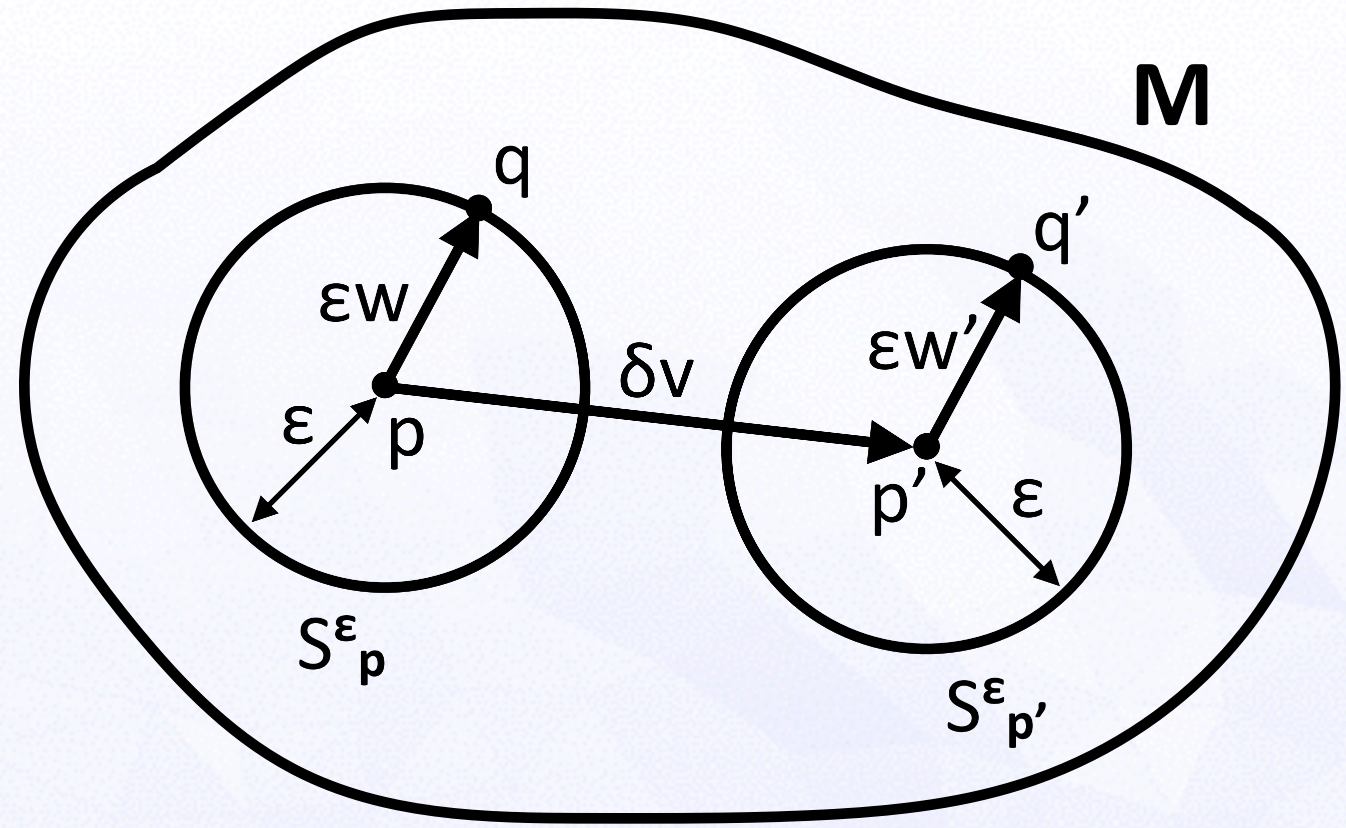
$\Rightarrow$ there is a unique point *$q' = exp_{p'}(\varepsilon w')$*

then we have $d(q,q') = \delta\Big(1 - \dfrac{\epsilon^2}{2}K(v,w) + O(\epsilon^3 + \delta\epsilon^2)\Big), \quad \epsilon, \delta \to 0$

where *$K(v,w)$* is the ***sectional curvature*** corresponding to *(v,w),*

$$K(v,w) = \frac{<R(v,w)w,v>}{<v,v><w,w> - <v,w>^2}$$

Step 2:



Let $S^{\varepsilon}{}_{p}$ denote the ε-sphere of all points at geodesic distance *ε* from its centre *p*. Then the ***sphere distance*** to another ε-sphere $S^{\varepsilon}{}_{p'}$ whose centre *p'* lies at a distance *d(p,p') = δ*, and whose points are obtained by parallel transport is defined by

$$d(S_p^\epsilon, S_{p'}^\epsilon) := \frac{1}{vol(S_p^\epsilon)} \int_{S_p^\epsilon} d^{D-1}q \, \sqrt{h} \, d(q, q'),$$

(note unique association q ↔ q')

which for small *δ, ε* is given by

$$d(S_p^\epsilon, S_{p'}^\epsilon) = \delta \left(1 - \frac{\epsilon^2}{2D} \, Ric(v, v) + O(\epsilon^3 + \delta\epsilon^2) \right),$$

where *Ric(v,v)* is the ***Ricci curvature*** of the vector *v*, i.e. the average of the sectional curvature *K(v,w)* over all two-planes containing *v*.

This formula,

$$d(S_p^\epsilon, S_{p'}^\epsilon) = \delta\left(1 - \frac{\epsilon^2}{2D}\, Ric(v,v) + O(\epsilon^3 + \delta\epsilon^2)\right),$$

captures our key statement: "On a *D*-dim. manifold with positive (negative) Ricci curvature, the distance between two nearby *(D-1)*-spheres is smaller (larger) than the distance between their centres."

This observation is the starting point for Ollivier's "coarse Ricci curvature" (Y. Ollivier, J. Funct. Anal. 256 (2009) 810-864).
[→ c.f. work by C. Trugenberger, G. Bianconi ("QG from graphs")]

However, his definition involves "transport distance" (in absence of parallel transport), which is very difficult to compute in practice.

Instead, we are looking for a notion of "quantum Ricci curvature" which is computable, scalable (also works for non-infinitesimal scales) and robust.

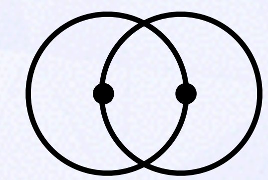
Defining quantum Ricci curvature

Our classical starting point is the ***average sphere "distance"***

$$\bar{d}(S_p^\epsilon, S_{p'}^\epsilon) := \frac{1}{vol(S_p^\epsilon)} \frac{1}{vol(S_{p'}^\epsilon)} \int_{S_p^\epsilon} d^{D-1}q \, \sqrt{h} \int_{S_{p'}^\epsilon} d^{D-1}q' \, \sqrt{h'} \, d(q, q'),$$

which uses only volume and distance measurements, and therefore can be implemented straightforwardly on general metric spaces.

We then set $\varepsilon = \delta$, corresponding to overlapping spheres, and define the **"quantum Ricci curvature K_q at scale δ"** by



$$\frac{\bar{d}(S_p^\delta, S_{p'}^\delta)}{\delta} = c_q(1 - K_q(p, p')), \quad \delta = d(p, p'), \quad c_q > 0,$$

where c_q appears to be a non-universal constant depending on the space under consideration. We have evaluated K_q on classical model spaces, and tested it on a variety of (mainly 2D) PL spaces (flat regular tilings, "nice" and "not-so-nice" triangulations).

Behaviour on constantly curved model spaces

Normalized <u>sphere distance</u> for 2D model spaces, for small *ε, δ:*

$$\frac{d(S_p^\epsilon, S_{p'}^\epsilon)}{\delta} = \begin{cases} 1 & \text{flat space} \\ 1 - \frac{1}{4}(\frac{\epsilon}{\rho})^2 + \text{h.o.} & \text{spherical case} \\ 1 + \frac{1}{4}(\frac{\epsilon}{\rho})^2 + \text{h.o.} & \text{hyperbolic case} \end{cases}$$

This is consistent with the general formula given above for *Ric =1/ρ²*.

Normalized <u>average sphere distance</u> for 2D model spaces, for *ε = δ* *and* small *ε, δ:*

$$\frac{\bar{d}(S_p^\delta, S_{p'}^\delta)}{\delta} = \begin{cases} 1.5746 & \text{flat space} \\ 1.5746 - 0.1440(\frac{\delta}{\rho})^2 + \text{h.o.} & \text{spherical case} \\ 1.5746 + 0.1440(\frac{\delta}{\rho})^2 + \text{h.o.} & \text{hyperbolic case} \end{cases}$$

This is qualitatively similar to the sphere distance results for *ε = δ.*

Behaviour on constantly curved model spaces

Normalized <u>sphere distance</u> for 2D model spaces, for small *ε, δ:*

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This is consistent with the general formula given above for $Ric = 1/\rho^2$.

Normalized <u>average sphere distance</u> for 2D model spaces, for *ε = δ and* small *ε, δ:*

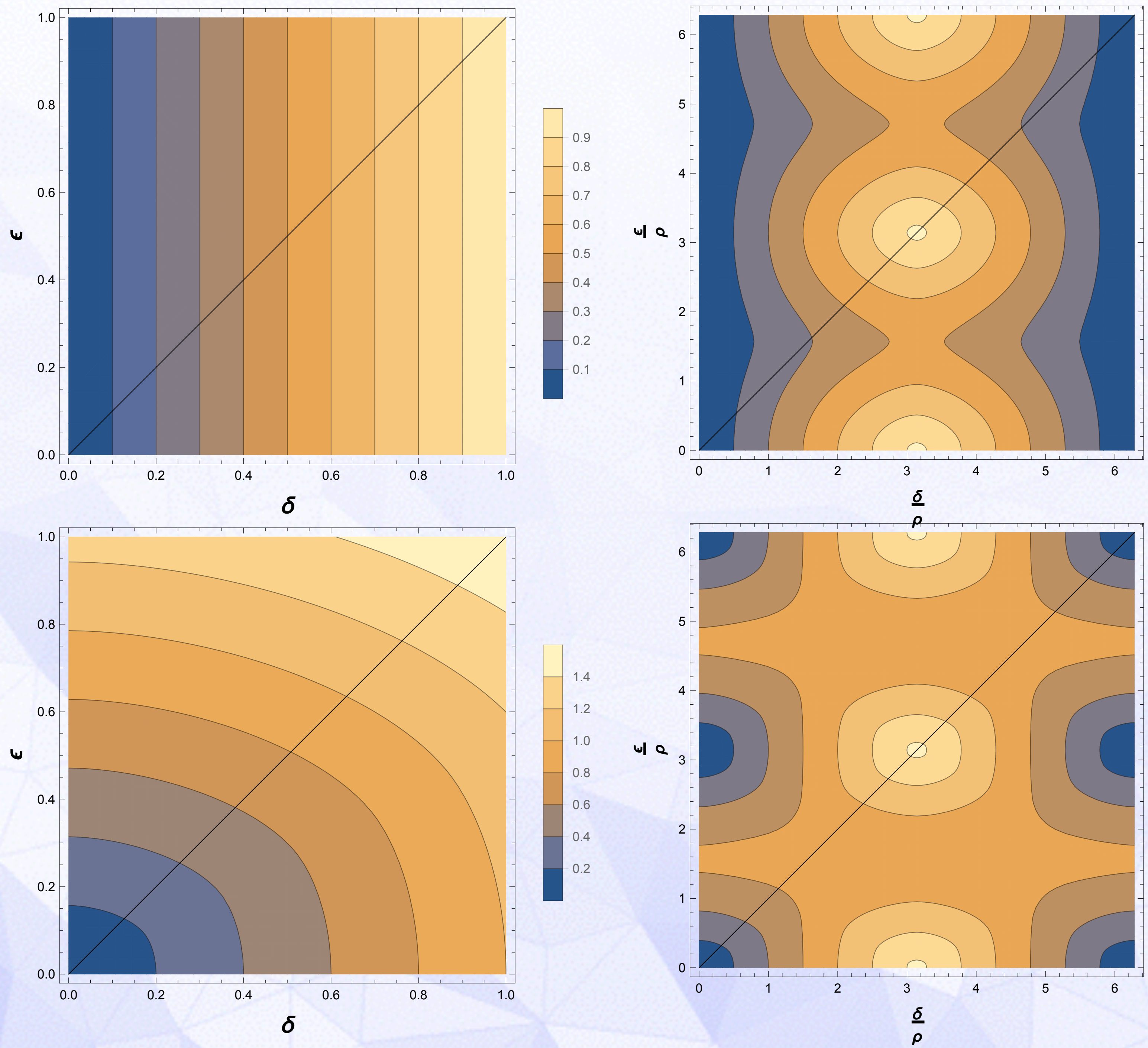
new

$$\frac{\bar{d}(S_p^\delta, S_{p'}^\delta)}{\delta} = \begin{cases} 1.5746 & \text{flat space} \\ 1.5746 - 0.1440(\frac{\delta}{\rho})^2 + \text{h.o.} & \text{spherical case} \\ 1.5746 + 0.1440(\frac{\delta}{\rho})^2 + \text{h.o.} & \text{hyperbolic case} \end{cases}$$

This is qualitatively similar to the sphere distance results for *ε = δ.*

Behaviour on constantly curved model spaces

flat case



spherical case

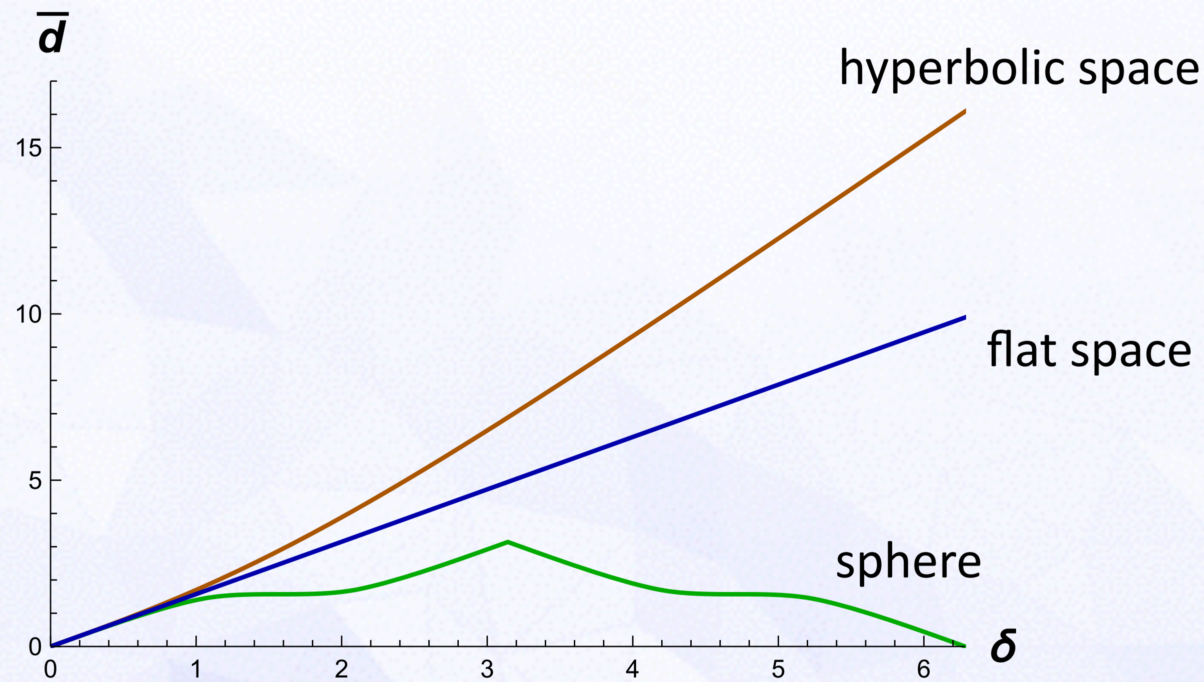
new

new

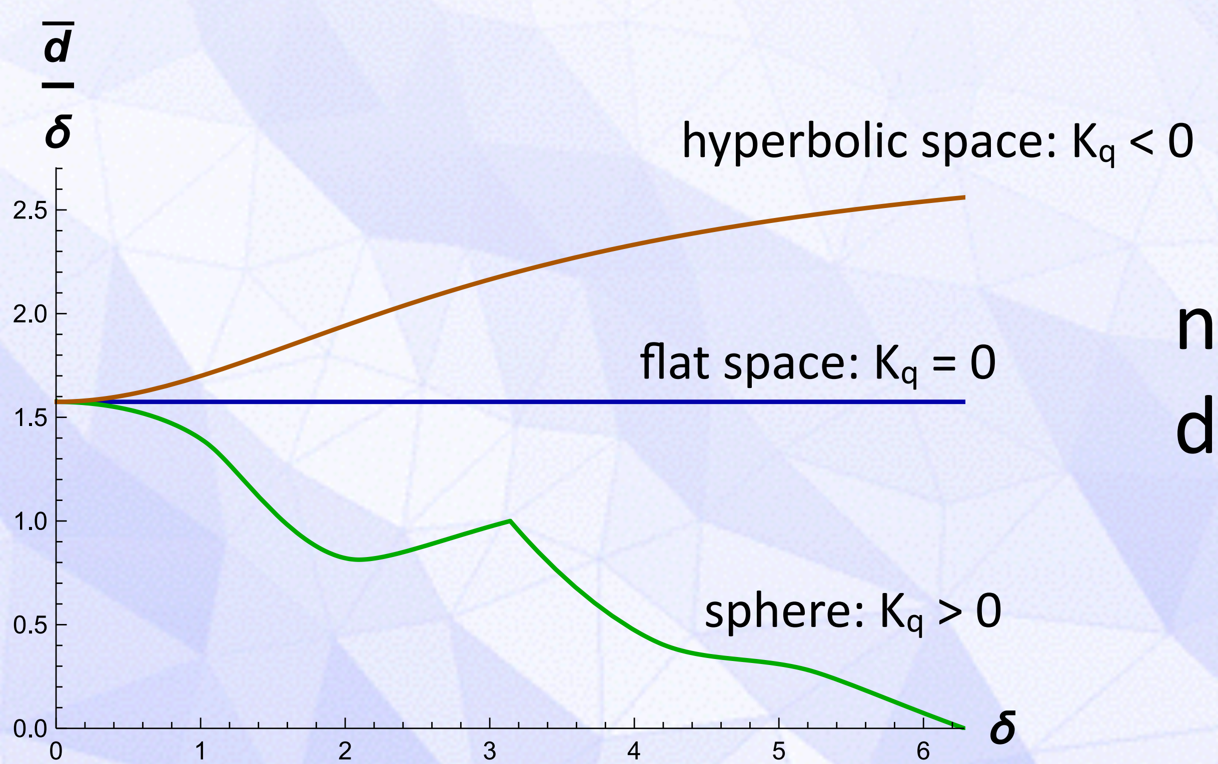
Comparing sphere distance and average sphere distance for any *(δ,ε)*.
Observe the qualitative similarities along the diagonals $\delta = \varepsilon$.

Behaviour on constantly curved model spaces

(after setting ε=δ)



average sphere distance $\bar{d}$ in the continuum (2D)

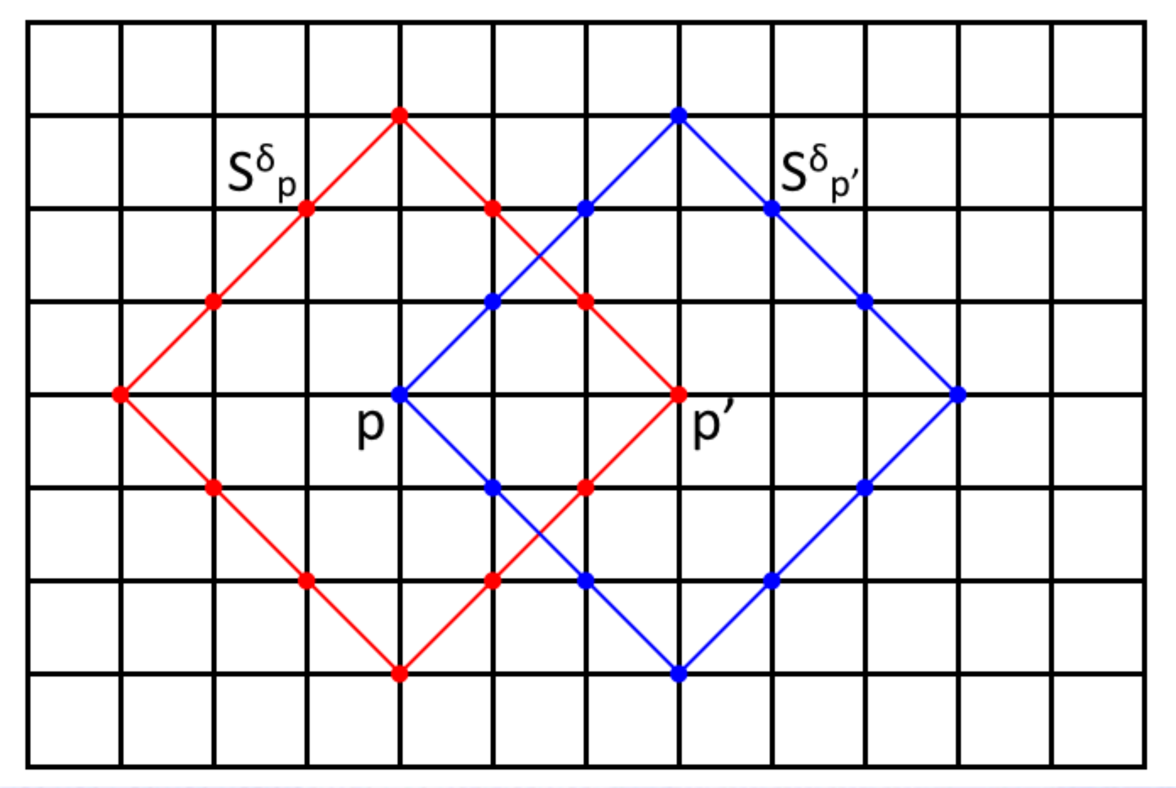


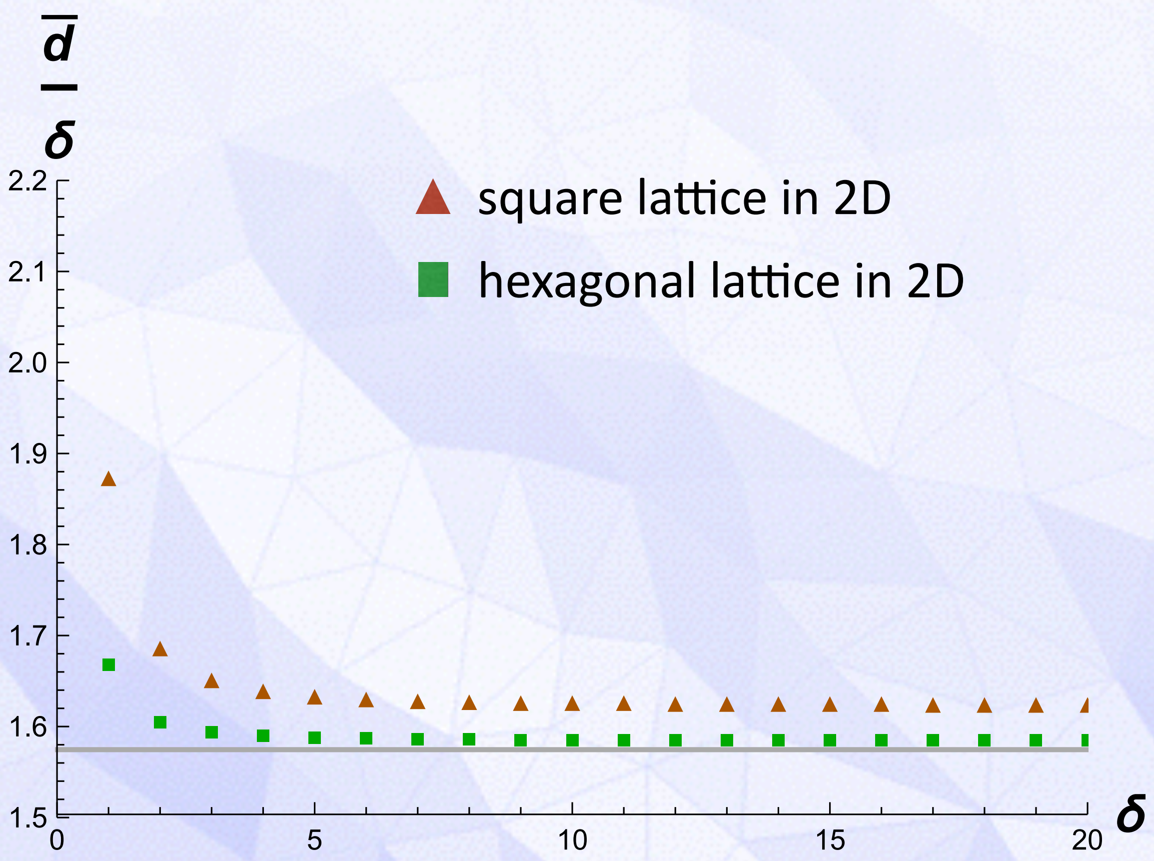
normalized average sphere distance $\bar{d}/\delta$ in the continuum (2D)

recall $\dfrac{\bar{d}}{\delta} = c_q(1 - K_q)$

"Nice" spaces I: regular triangulations

Example: sphere distance on a 2D square lattice (*δ=3*)





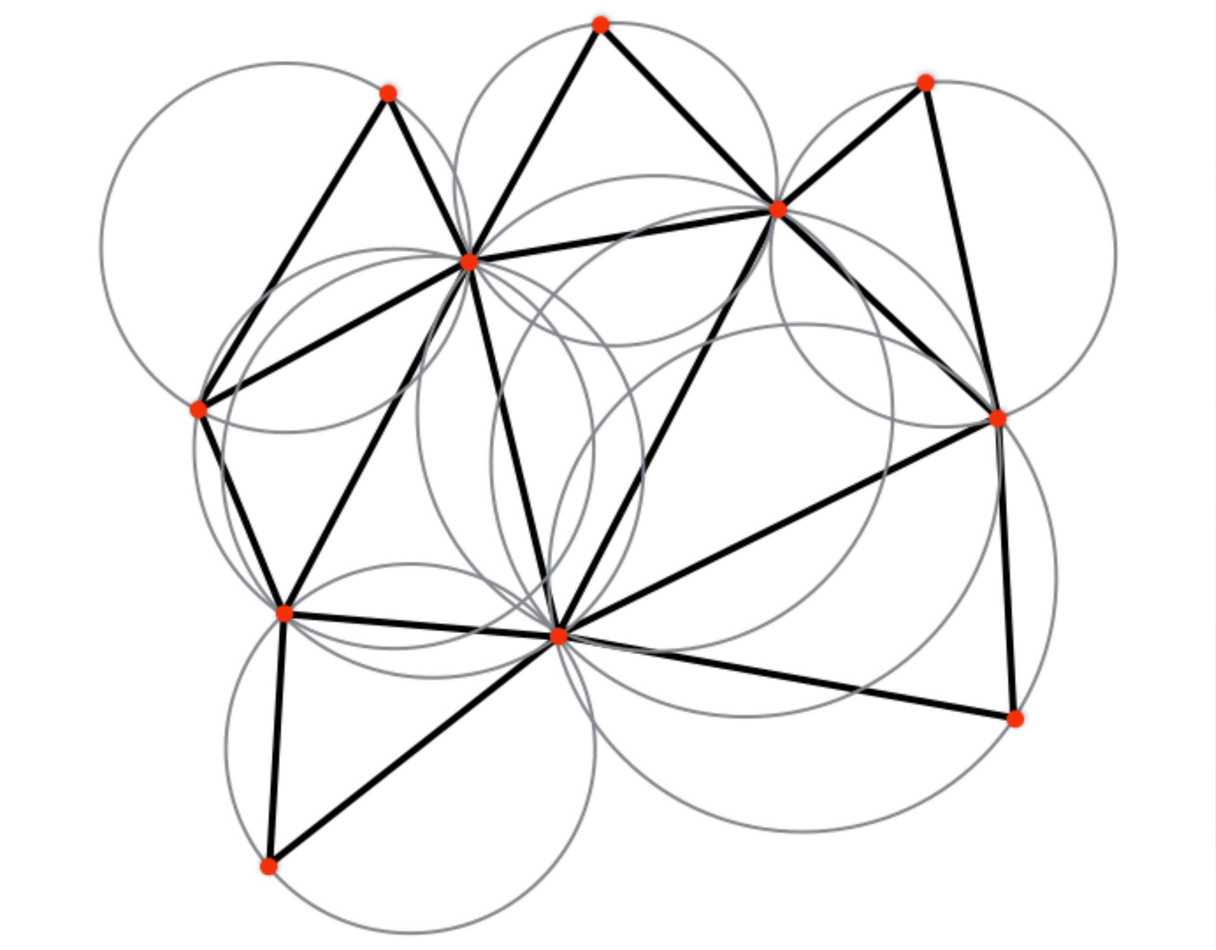
We measured normalized average sphere distances $\bar{d}/\delta$ for regular flat lattices in 2D and 3D.

Their typical behaviour is $\bar{d}/\delta = c_q + \frac{1}{\delta^2}$-corrections ($c_q$ not universal).

How can we construct equilateral random geometries that are “close” to a given smooth classical geometry?

"Nice" spaces II: Delaunay triangulations

A Delaunay triangulation is a triangulation T of a finite point set $P \subset \mathbf{R}^2$ (the vertices of T) if the circumcircle of every triangle contains no points of P in its interior. It maximizes the minimum angle and avoids thin, elongated triangles



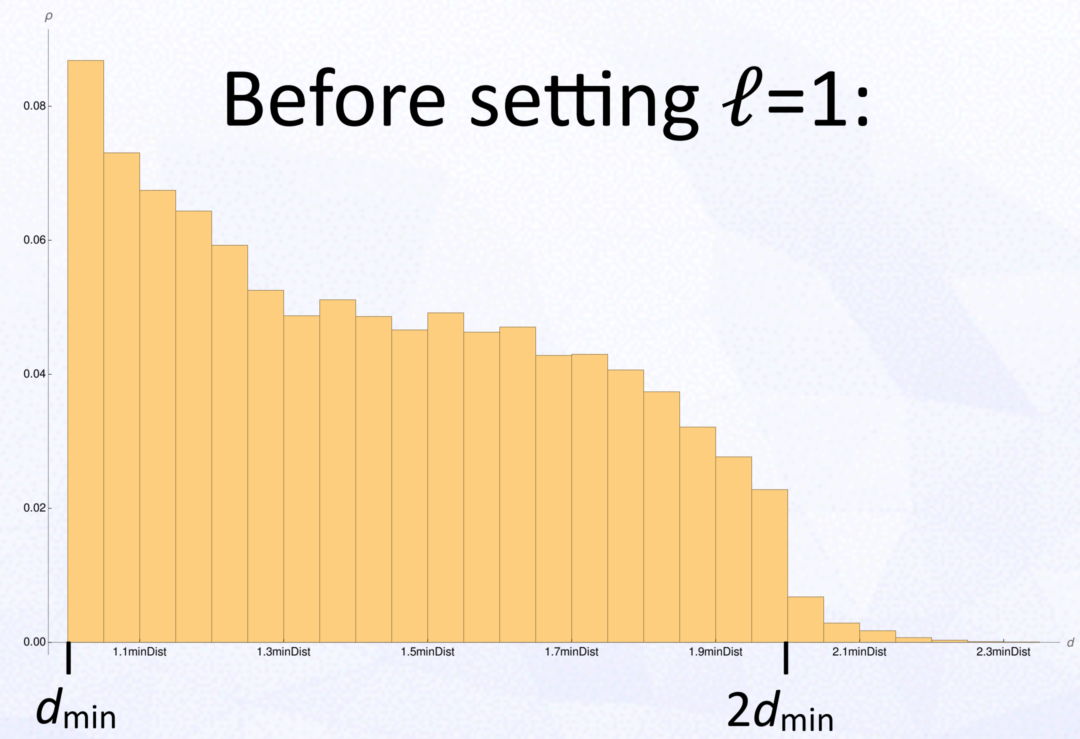
Our procedure:

1. generate P using Poisson disc sampling on a 2D constantly curved space (plane, sphere, hyperboloid)
2. construct the Delaunay triangulation of P
3. set all edge lengths to 1

How nice are the resulting PL spaces?

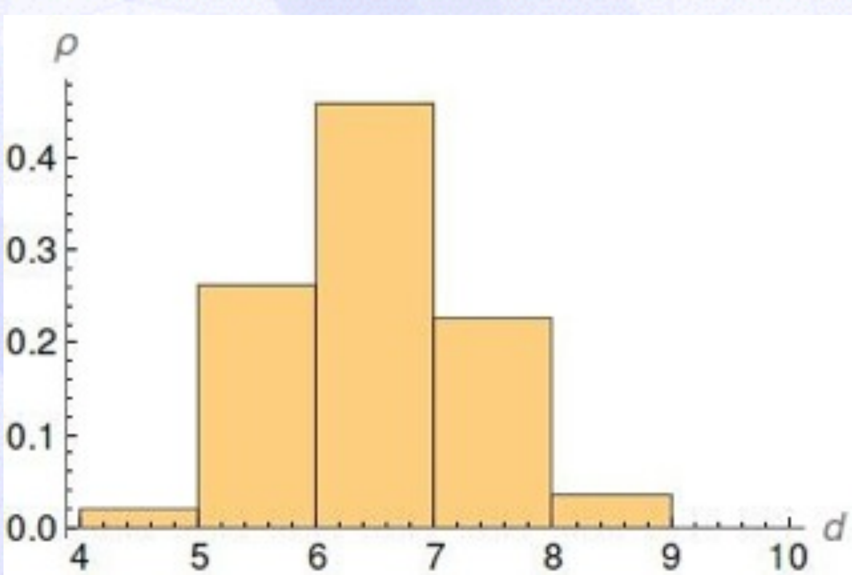
Our construction generates random triangulations with mild (small-scale) local curvature fluctuations.

Before setting ℓ=1:



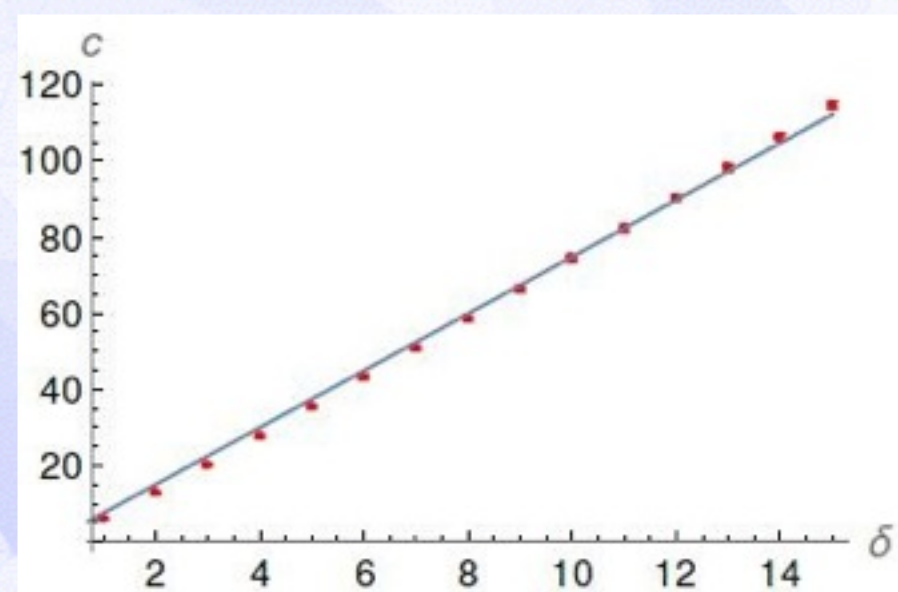
probability distribution of edge lengths ℓ in a Delaunay triangulation of flat space, *N=6.2k*

Before and after:



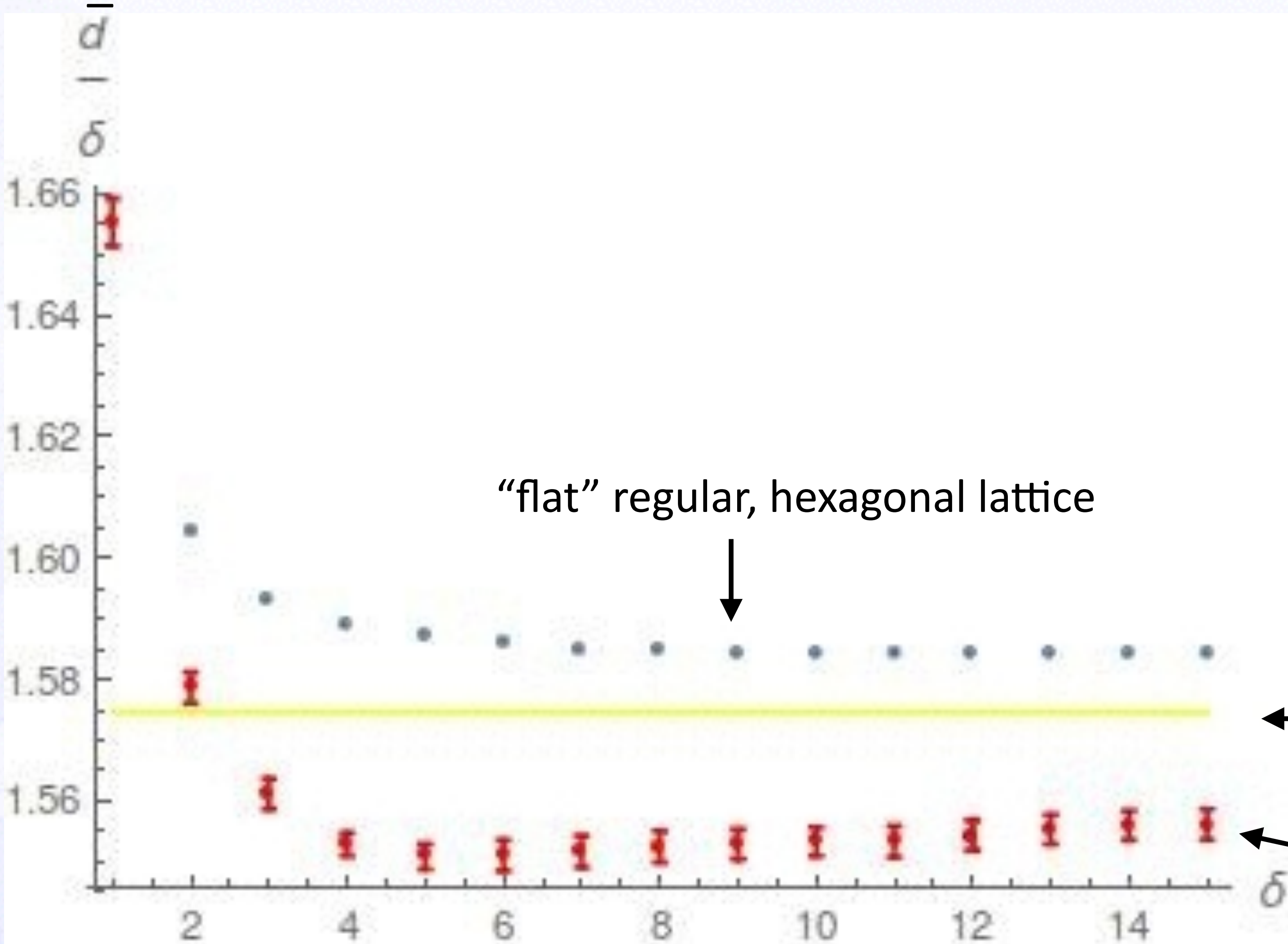
distribution of vertex orders (flat case)

After:



volume *c* of geodesic circles of radius *δ:* linearity implies flat-space behaviour

Measuring the quantum Ricci curvature K_q



flat continuum space

"flat" random triangulation

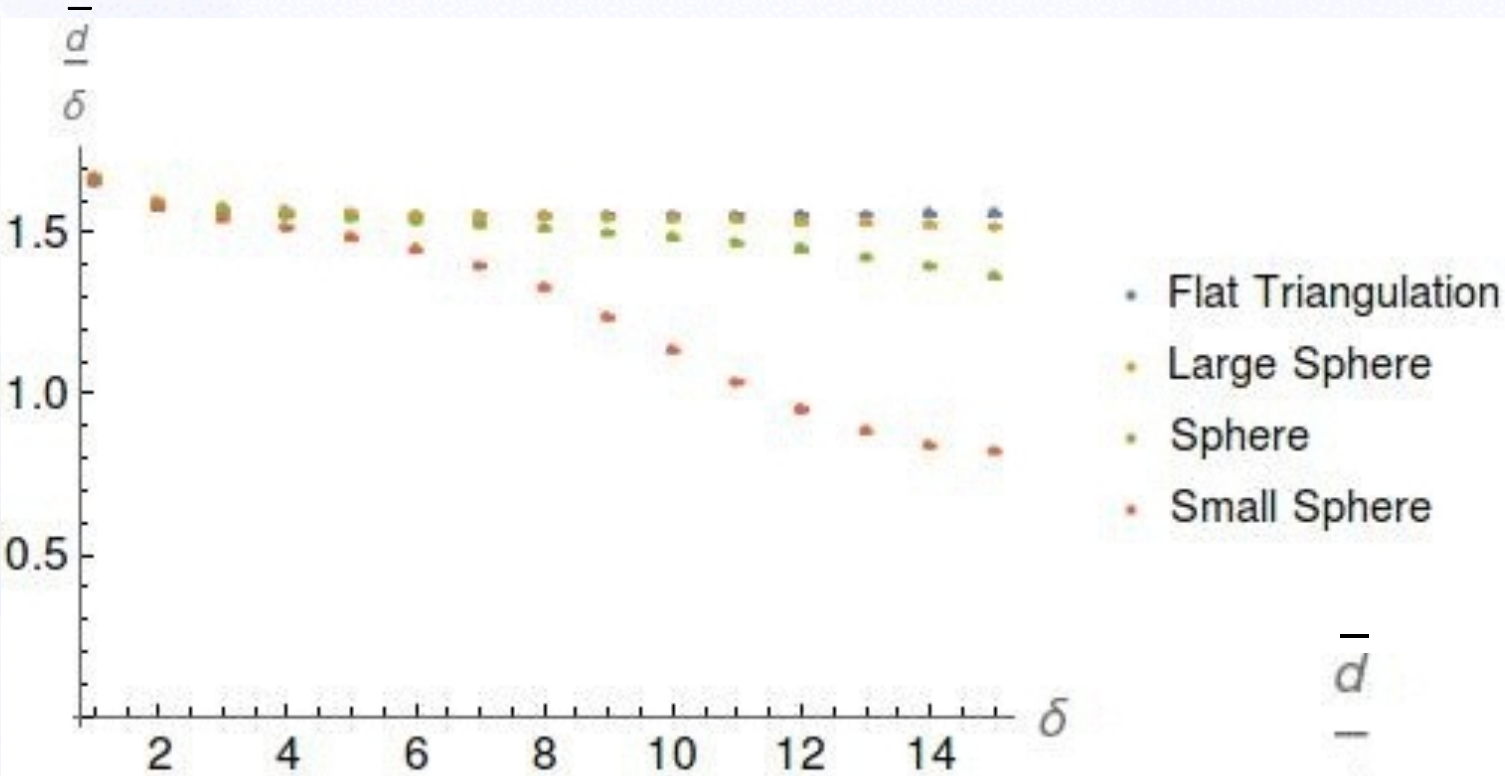
comparison of measurements of the normalized average sphere distance $\bar{d}/\delta$

recall that

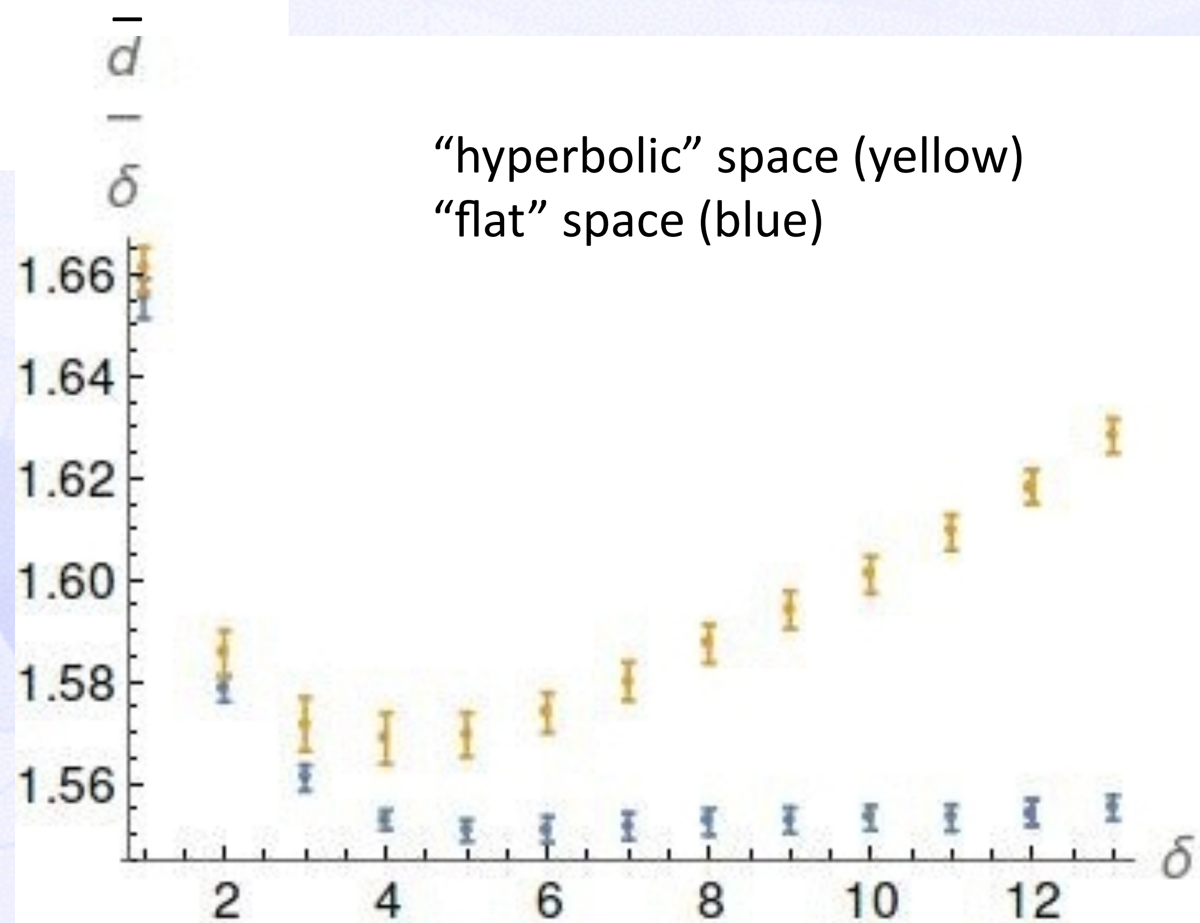
$$\bar{d}/\delta = c_q(1 - K_q(\delta))$$

We obtain a good matching with continuum results, with discretization artefacts confined to the region $\delta \lesssim 5$, and $K_q \approx 0$ elsewhere.

Measuring the quantum Ricci curvature, ctd.



normalized average sphere distance $\bar{d}/\delta$ for random triangulations modelled on flat space and spheres of various sizes



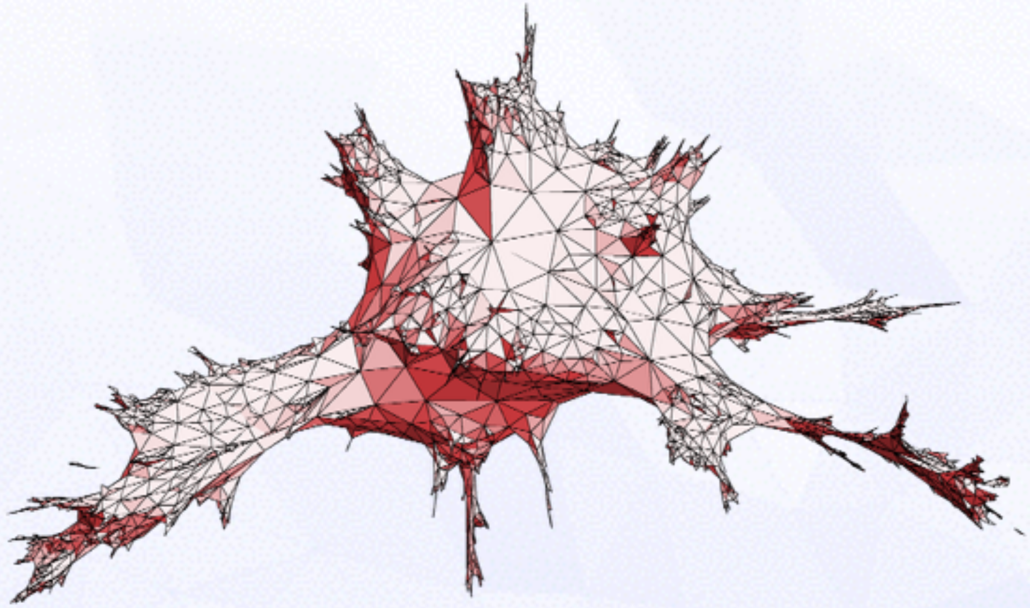
normalized average sphere distance $\bar{d}/\delta$ for random triangulations modelled on flat and hyperbolic space

We observe good “averaging properties”.

A true *quantum* application of Ricci curvature

- consider a 2D toy model of (Euclidean) quantum gravity, with

$$Z(\Lambda) = \int_{\mathrm{geom.}\, g} \mathcal{D}g \; \mathrm{e}^{-\Lambda\, vol(g)}$$



a typical "universe" in DT quantum gravity in 2D

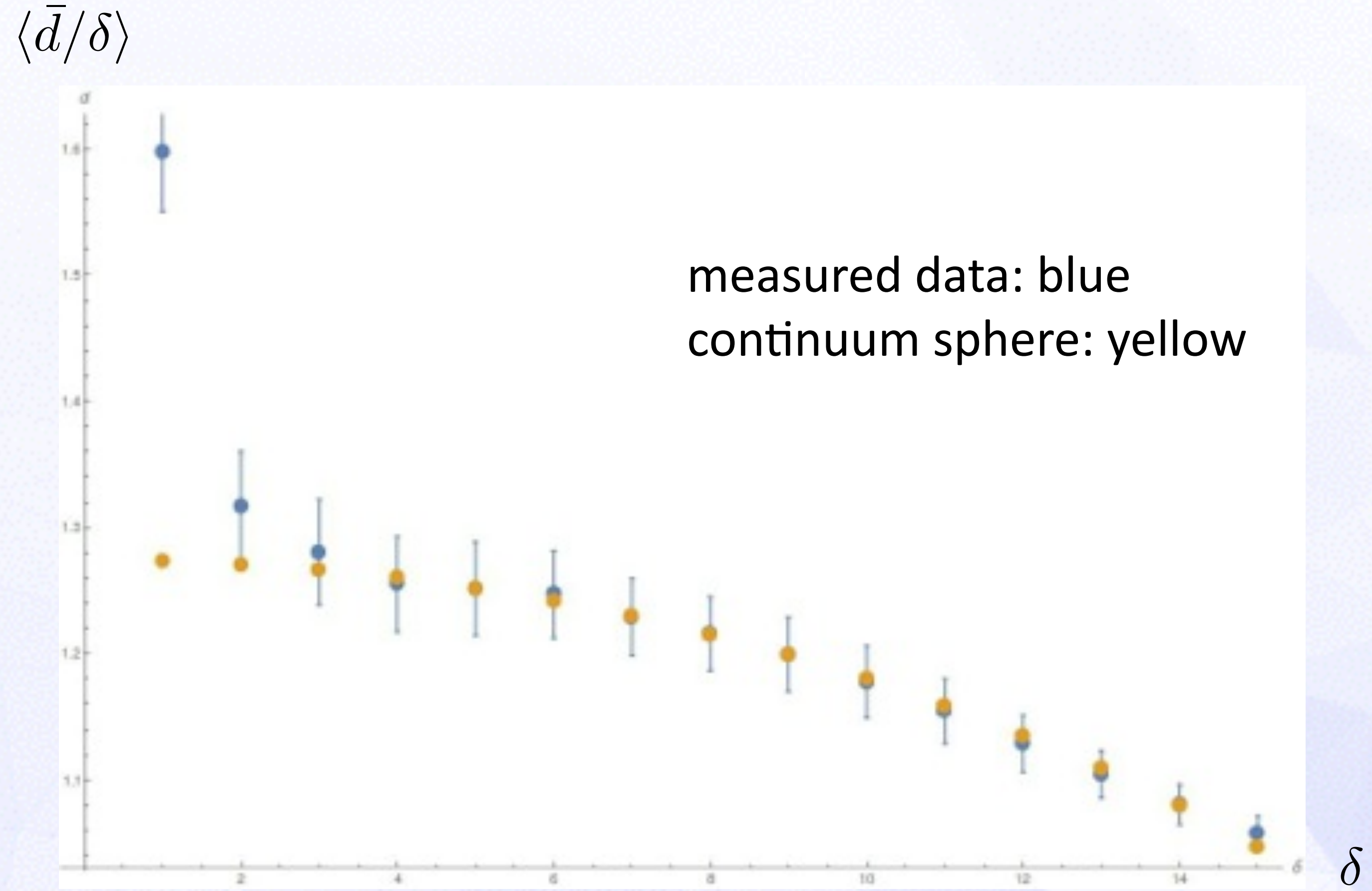
- nonperturbative path integral over geometries with fixed topology S^2, soluble via "Dynamical Triangulations"
- path integral configurations are arbitrary gluings of 2D equilateral triangles; in the continuum limit, "typical" ensemble members are highly nonclassical, fractal and nowhere differentiable geometries with spectral dimension 2 and Hausdorff dimension 4
- the quantum dynamics is governed by branching "baby universes"

Quantum Ricci curvature in 2D DT QGravity

We have measured the expectation value

$$\langle \bar{d}(S_p^\delta, S_{p'}^\delta)/\delta \rangle$$

of the normalized average sphere distance in the ensemble of 2D Euclidean DT quantum gravity.



Using a system of 20.000 triangles, we found a surprisingly good fit of the data with those of a continuum sphere in 4D(!), with positive curvature. This points to a *very robust behaviour of quantum Ricci curvature*. (N.B.: the S^δ_p in general do not even have spherical topology for *δ>1*, but are multiply connected.)

Summary

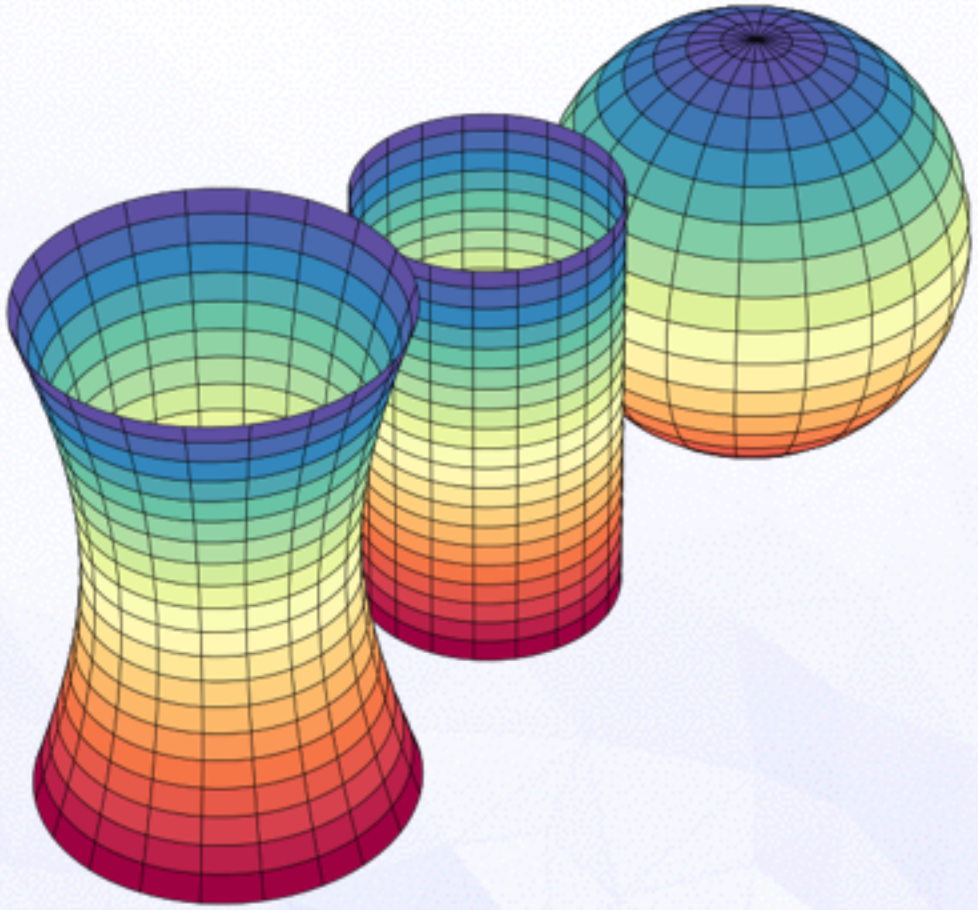
We have defined a novel notion of "*quantum Ricci curvature at scale δ*", based on the average sphere distance $\bar{d}(S_p^\delta, S_{p'}^\delta)/\delta$, and have investigated its properties in both a classical and quantum context, mostly on two-dimensional geometries. The results look promising:

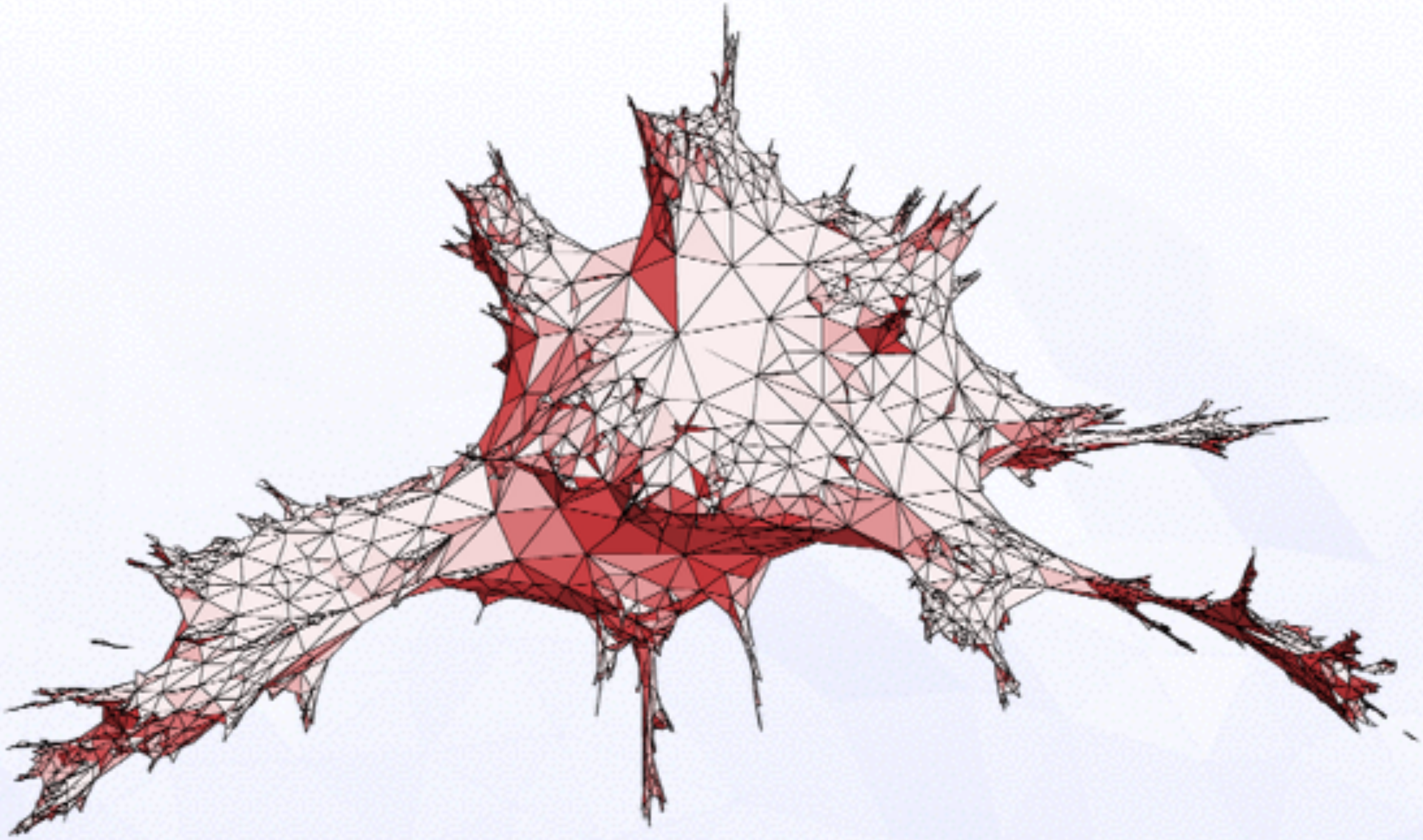
• the prescription is straightforward to implement on piecewise flat spaces and is feasible computationally;

• on nice piecewise flat spaces, lattice artefacts can be controlled and smooth results are reproduced on sufficiently large scales;

• "robustness" has been found in the case of the highly quantum-fluctuating quantum ensemble of 2D Euclidean DT quantum gravity.

The next step is an implementation in 4D Causal Dynamical Triangulations, a nonperturbative candidate theory of quantum gravity, to understand and quantify its quantum geometry.





Thank you!