Piecewise Flat Quantum Gravity

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Path integral for GR

- $M$ a smooth 4-manifold, $g$ a metric on $M$ then

$$Z(M) = \int \mathcal{D}g \exp \left( iS_{AH}(g, M)/\hbar \right) = ?$$

- Regge: $T(M)$ a triangulation of $M$, replace $g$ by $\{L_\epsilon \mid \epsilon \in T_1(M)\}$ and

$$Z(M) = \lim_{T(M) \to M} Z(T(M)).$$

- It is very difficult to find a value of $Z$ in the smooth limit $T(M) \to M$. 
Assume that $T(M)$ is the fundamental spacetime structure, i.e. the spacetime is a *piecewise linear* 4-manifold $T(M)$ with a flat metric in each cell (4-symplex). If $N$ is the number of cells of $T(M)$, then for $N \gg 1$, $T(M)$ looks like the smooth manifold $M$.

$Z(T(M))$ can be made finite and it is not necessary to define the smooth limit $N \to \infty$. Instead, we need the large-$N$ approximation for the observables. Analogous to the fluid dynamics situation.

The semiclassical limit will be described by the effective action $\Gamma(L)$, which is computed by using the effective action equation from QFT, in the limit $L_c \gg l_P = \sqrt{G_N \hbar}$.
Regge state-sum model

- The fundamental DOF are the edge-lengths $L_\epsilon \geq 0$, and

$$Z = \int_{D_E} \prod_{\epsilon=1}^E dL_\epsilon \mu(L) \exp \left( i S_{Rc}(L)/l_P^2 \right),$$

where

$$S_{Rc} = -\sum_{\Delta=1}^F A_\Delta(L) \theta_\Delta(L) + \Lambda_c V_4(L),$$

is the Regge action with a cosmological constant. $D_E \subset \mathbb{R}_+^E$ such that the triangle inequalities hold.

- We choose the following PI measure

$$\mu(L) = e^{-V_4(L)/L_0^4},$$

where $L_0$ is a free parameter.

- We also introduce a classical CC length scale $L_c$, $\Lambda_c = \pm 1/L_c^2$ ⇒ $L_c$ is the second free parameter.
Effective action vs Wilsonian approach

- We will use the *effective action* in order to determine the quantum corrections.
- The EA approach is different from the *Wilsonian* approach to quantization which is used in Quantum Regge Calculus and in Casual Dynamical Triangulations.
- In the Wilsonian approach

\[ Z(\kappa, \lambda) = \int_{D_E} \prod_{\epsilon=1}^{E} dL_\epsilon \mu(L) \exp \left[ i\kappa S_R(L)/l_0^2 + i\lambda V_4(L)/l_0^4 \right], \]

and one looks for the points \((\kappa, \lambda)\) where \(Z''\) diverges.
- In the vicinity of a critical point the correlation length diverges \(\Leftrightarrow\) transition from the discrete (finitely many DOF) to a continuum (infinitely many DOF) theory.
- The semiclassical limit \((l_P \to 0)\) in WA corresponds to the strong-coupling limit \(\kappa \to \infty\), hence it can be only studied numerically.
Let $\phi : M \to \mathbb{R}$ and $S(\phi) = \int_M d^4x \, \mathcal{L}(\phi, \partial \phi)$ a QFT flat-spacetime action. The effective action $\Gamma(\phi)$ is determined by the integro-differential equation

$$e^{i\Gamma(\phi)/\hbar} = \int \mathcal{D}h \exp \left[ \frac{i}{\hbar} S(\phi + h) - \frac{i}{\hbar} \int_M d^4x \, \frac{\delta \Gamma}{\delta \phi(x)} h(x) \right].$$

A generic solution $\Gamma(\phi) \in \mathbb{C}$. Wick rotation is used to obtain $\Gamma(\phi) \in \mathbb{R}$: solve the Euclidean-space equation

$$e^{-\Gamma_E(\phi)/\hbar} = \int \mathcal{D}h \exp \left[ -\frac{1}{\hbar} S_E(\phi + h) + \frac{1}{\hbar} \int_M d^4x \, \frac{\delta \Gamma_E}{\delta \phi(x)} h(x) \right],$$

and then put $x_0 = -it$ in $\Gamma_E(\phi)$.

Wick rotation is equivalent to $\Gamma(\phi) \to \text{Re} \, \Gamma(\phi) + i \text{Im} \, \Gamma(\phi)$. 
Regge effective action

- In the case of a Regge state-sum model

\[ e^{i\Gamma(L)/l_P^2} = \int_{D_E(L)} d^E x \mu(L + x) e^{iS_{Rc}(L+x)/l_P^2 - i \sum_{\epsilon=1}^{E} \Gamma'(L) x_\epsilon / l_P^2}, \]

where \( l_P^2 = G_N \hbar \) and \( D_E(L) \) is a subset of \( \mathbb{R}^E \) obtained by translating \( D_E \) by a vector \(-L\).

- \( D_E(L) \subseteq [-L_1, \infty) \times \cdots \times [-L_E, \infty) \).

- Semiclassical solution

\[ \Gamma(L) = S_{Rc}(L) + l_P^2 \Gamma_1(L) + l_P^4 \Gamma_2(L) + \cdots, \]

where \( L_\epsilon \gg l_P \) and

\[ |\Gamma_n(L)| \gg l_P^2 |\Gamma_{n+1}(L)|. \]
Perturbative solution

Let $L_\epsilon \to \infty$, then $D_E(L) \to \mathbb{R}^E$ and

$$e^{i\Gamma(L)/l_P^2} \approx \int_{\mathbb{R}^E} d^E x \, \mu(L + x) e^{iS_{Re}(L+x)/l_P^2 - i \sum_{\epsilon=1}^E \Gamma'_\epsilon(L)x_\epsilon/l_P^2}.$$ 

The reason is $D_E(L) \approx [-L_1, \infty) \times \cdots \times [-L_E, \infty)$ so that

$$\int_{-L}^\infty dx \, e^{-zx^2/l_P^2 - w x} = \sqrt{\pi} \, l_P \exp \left[ -\frac{1}{2} \log z + l_P^2 \frac{w^2}{4z} \right. + l_P \frac{e^{-z\bar{L}^2/l_P^2}}{2\sqrt{\pi z \bar{L}}} \left(1 + O(l_P^2/z\bar{L}^2))\right],$$

where $\bar{L} = L + l_P^2 \frac{w}{2z}$ and $Re \, z = -(\log \mu)''$. The non-analytic terms in $\hbar$ will be absent if

$$\lim_{L \to \infty} e^{-z\bar{L}^2/l_P^2} = 0 \iff (\log \mu)'' < 0 \text{ for } L \to \infty.$$

Hence the perturbative solution exists for the exponentially damped measures.
Perturbative solution

- For $D_E(L) = R^E$ and $\mu(L) = const.$ the perturbative solution is given by the EA diagrams

$$\Gamma_1 = \frac{i}{2} Tr \log S''_{Rc}, \quad \Gamma_2 = \langle S_3^2 G^3 \rangle + \langle S_4 G^2 \rangle,$$

$$\Gamma_3 = \langle S_3^4 G^6 \rangle + \langle S_3^2 S_4 G^5 \rangle + \langle S_3 S_5 G^4 \rangle + \langle S_4^2 G^4 \rangle + \langle S_6 G^3 \rangle, \ldots$$

where $G = i(S''_{Rc})^{-1}$ is the propagator and $S_n = iS^{(n)}_{Rc}/n!$ for $n > 2$, are the vertex weights.

- When $\mu(L) \neq const.$, the perturbative solution is given by

$$\Gamma(L) = \tilde{S}_{Rc}(L) + l_P^2 \tilde{\Gamma}_1(L) + l_P^4 \tilde{\Gamma}_2(L) + \cdots,$$

where

$$\tilde{S}_{Rc} = S_{Rc} - il_P^2 \log \mu,$$

while $\tilde{\Gamma}_n$ is given by the sum of $n$-loop EA diagrams with $\tilde{G}$ propagators and $\tilde{S}_n$ vertex weights.
Therefore

\[ \Gamma_1 = -i \log \mu + \frac{i}{2} \text{Tr} \log S'_{Rc} \]

\[ \Gamma_2 = \langle S_3^2 G^3 \rangle + \langle S_4 G^2 \rangle + \text{Res}[l_P^{-4} \text{Tr} \log \bar{G}] , \]

\[ \Gamma_3 = \langle S_3^4 G^6 \rangle + \cdots + \langle S_6 G^3 \rangle + \text{Res}[l_P^{-6} \text{Tr} \log \bar{G}] + \text{Res}[l_P^{-6} \langle \bar{S}_3^2 \bar{G}^3 \rangle] + \text{Res}[l_P^{-6} \langle \bar{S}_4 \bar{G}^2 \rangle] , \cdots \]

Therefore

\[ \Gamma_1(L) = O((L/L_0)^4) \quad \text{and for} \quad L_\epsilon > L_c , \quad L_0 > \sqrt{l_P L_c} \]

we get the following large-\( L \) asymptotics

\[ \Gamma_1(L) = O(L^4/L_0^4) + \log O(L^2/L_c^2) + \log \theta(L) + O(L_c^2/L^2) \]

and

\[ \Gamma_{n+1}(L) = O ((L_c^2/L^4)^n) + L_0^{-2n} O ((L_c^2/L^2)) , \]

where \( L_0c = L_0^2/L_c \).
For $L_\epsilon \gg l_P$ and $L_0 \gg \sqrt{l_P L_c}$ the series

$$\sum_{n \geq 0} (l_P)^{2n} \Gamma_n(L)$$

is semiclassical.

Let $\Gamma \rightarrow \Gamma / G_N$ so that $S_{\text{eff}} = (\text{Re}\, \Gamma \pm \text{Im}\, \Gamma) / G_N$

One-loop CC

$$S_{\text{eff}} = \frac{S_{Rc}}{G_N} \pm \frac{l_P^2}{G_N L_0^4} V_4 \pm \frac{l_P^2}{2 G_N} \text{Tr} \log S''_{Rc} + O(l_P^4) \Rightarrow$$

$$\Lambda = \Lambda_c \pm \frac{l_P^2}{2 L_0^4} = \Lambda_c + \Lambda_{qg}.$$ 

The one-loop cosmological constant is exact because there are no $O(L^4)$ terms beyond the one-loop order.
This is a consequence of the large-$L$ asymptotics

\[
\log S''_{Rc}(L) = \log O(L^2/\bar{L}_c^2) + \log \theta(L) + O(\bar{L}_c^2/L^2)
\]

\[
\bar{\Gamma}_{n+1}(L) = O((\bar{L}_c^2/L^4)^n),
\]

where \( \bar{L}_c^2 = L_c^2 [1 + i l_P^2 (L_c^2/L_0^4)]^{-1/2} \).

Note that \( L_c \geq l_P \) implies \( L_0 \gg l_P \), so that

\[
l_P^2 |\Lambda_{qg}| = \frac{1}{2} \left( \frac{l_P}{L_0} \right)^4 \ll 1.
\]

If \( L_c < l_P \) then \( L_0 \gg l_P \) is consistent with the semiclassical approximation.

If \( \Lambda_c = 0 \), the observed value of \( \Lambda \) is obtained for \( L_0 \approx 10^{-5} m \) so that \( l_P^2 \Lambda \approx 10^{-122} \).
Smooth-spacetime limit

- Smooth spacetime is described by $T(M)$ with $E \gg 1 \Rightarrow$
  
  $$S_R(L) \approx \frac{1}{2} \int_M d^4x \sqrt{|g|} R(g),$$
  
  $$\Lambda V_4(L) \approx \int_M d^4x \sqrt{|g|} = \Lambda V_M,$$

- For $L_\epsilon \geq L_K \gg l_P$ and $E \gg 1$

  $$Tr \log S''_R(L) \approx \int_M d^4x \sqrt{|g|} \left[ a(L_K)R^2 + b(L_K)R_{\mu\nu}R^{\mu\nu} \right].$$

- When $M = \Sigma \times I$, $L_K$ defines a QFT momentum UV cutoff $\hbar/L_K$. LHC experiments $\Rightarrow L_K < 10^{-20} \text{m} \iff \hbar K > 10 \text{ TeV}$. 
Coupling of matter

- Scalar field on \( M \)

\[
S_m(g, \phi) = \frac{1}{2} \int_M d^4x \sqrt{|g|} \left[ \phi^\mu \partial_\mu \phi \partial_\nu \phi - U(\phi) \right],
\]

where \( U = \frac{1}{2} \omega^2 \phi^2 + \lambda \phi^4 \).

- On \( T(M) \) we get

\[
S_m = \frac{1}{2} \sum_\sigma V_\sigma(L) \sum_{k,l} g_{kl}^\sigma(L) \phi'_k \phi'_l - \frac{1}{2} \sum_\pi V^*_\pi(L) U(\phi_\pi),
\]

where \( \phi'_k = (\phi_k - \phi_0)/L_{0k} \).

- The total classical action

\[
S(L, \phi) = \frac{1}{G_N} S_{Rc}(L) + S_m(L, \phi).
\]
Coupling of matter

- The EA equation

\[ e^{i \Gamma(L, \phi)/l_P^2} = \int_{D_E(L)} d^E x \int_{\mathbb{R}^V} d^V \chi \exp \left[ i \tilde{S}_{Rm}(L + x, \phi + \chi)/l_P^2 \right. \]

\[-i \sum_\epsilon \frac{\partial \Gamma}{\partial L_\epsilon} x_\epsilon/l_P^2 - i \sum_\pi \frac{\partial \Gamma}{\partial \phi_\pi} \chi_\pi/l_P^2 \],

where \( \tilde{S}_{Rm} = \tilde{S}_{Rc} + G_N S_m(L, \phi) \).

- Perturbative solution

\[ \Gamma(L, \phi) = S(L, \phi) + l_P^2 \Gamma_1(L, \phi) + l_P^4 \Gamma_2(L, \phi) + \cdots \]

is semiclassical for \( L_\epsilon \gg l_P, L_0 \gg l_P \) and \( |\sqrt{G_N} \phi| \ll 1 \). This can be checked in \( E = 1 \) toy model

\[ S(L, \phi) = (L^2 + L^4/L_c^2) \theta(L) + L^2 \theta(L) \phi^2 (1 + \omega^2 L^2 + \lambda \phi^2 L^2) \].

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Coupling of matter

- \( \Gamma(L, \phi) = \Gamma_g(L) + \Gamma_m(L, \phi) \)
- \( \Gamma_m(L, \phi) = V_4(L) U_{\text{eff}}(\phi) \) for constant \( \phi \) and \( U_{\text{eff}}(0) = 0 \).
- \( \Gamma_g(L) = \Gamma_{pg}(L) + \Gamma_{mg}(L) \) and

\[
\Gamma_{mg}(L) \approx \Lambda_m V_M + \Omega_m(R, K)
\]

in the smooth-manifold approximation and \( K = 1/L_K \).
- \( \Omega_m = \Omega_1 l_p^2 + O(l_p^4) \) and

\[
\Omega_1(R, K) = a_1 K^2 \int_M d^4x \sqrt{|g|} R^+
\]

\[
\log(K/\omega) \int_M d^4x \sqrt{|g|} \left[ a_2 R^2 + a_3 R^{\mu\nu} R_{\mu\nu} + a_4 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + a_5 \nabla^2 R \right] + O(L_K^2/L^2) .
\]
The effective CC

\[ \Lambda = \Lambda_c + \Lambda_{qg} + \Lambda_m, \]

where

\[ \Lambda_m \approx \sum_{\gamma} \nu(\gamma, K) \]

and \( \nu(\gamma, K) \) is a 1PI vacuum FD for \( S_m \) with the cutoff \( K \).

For \( K \gg \omega \)

\[ \sum_{\gamma} \nu(\gamma, K) \approx l_P^2 K^4 \left[ c_1 \ln(K^2/\omega^2) + \sum_{n \geq 2} c_n (\bar{\lambda})^{n-1} (\ln(K^2/\omega^2))^{n-2} \right. \]

\[ + \left. \sum_{n \geq 4} d_n (\bar{\lambda})^{n-1} (K^2/\omega^2)^{n-3} \right] , \]

where \( \bar{\lambda} = l_P^2 \lambda \).

In QFT \( \Lambda_m = \lim_{K \to \infty} \sum_{\gamma} \nu(\gamma, K) = \infty \). Not clear how to define a NP value.
In PLQG the exact solution of the EA equation will give a finite and cutoff-independent value

$$\Lambda_m = V(\omega^2, \lambda, l_P^2).$$

Hence

$$\Lambda = \pm \frac{1}{L_c^2} + \frac{l_P^2}{2L_0^4} + V(\omega^2, \lambda, l_P^2).$$

We can fix the free parameter $L_c$ by setting

$$\Lambda_c + \Lambda_m = 0,$$

while $L_0$ can be fixed by matching $\Lambda$ to the observed CC value:

$$\Lambda = l_P^2/2L_0^4 \Rightarrow L_0 \approx 10^{-5} \text{ m}.$$
The CC problem in QG has two parts (Polchinski 2006):

1. Show that the observed CC value is in the CC spectrum.
2. Explain why CC takes the observed value.

PLQG solves (1).

String theory may solve (1): CC spectrum is discrete with $10^{500}$ values (Russo and Polchinski 2000) and positive values are allowed (KKLT 2003) ⇒ plausible to assume that the CC spectrum is sufficiently dense around zero.

(2) is a much harder question, and one "explanation" is the anthropic principle.
Quantum cosmology

- The EA formalism is only applicable for $M = \Sigma \times I$.
- In the case $M \neq \Sigma \times I$, $\partial M = \Sigma$, one can define a Hartle-Hawking wf of the Universe
  \[ \Psi_0(L_s) = Z(T(M)) \text{ with } L|_{T(\Sigma)} = L_s. \]
- Big Bang $\Leftrightarrow M = M_0 \cup (\Sigma \times I)$.
- Cosmological bounce $\Leftrightarrow M = \Sigma \times \mathbb{R}$.
- Hamiltonian evolution on $T(\Sigma \times I)$ possible if
  \[ T_1(\Sigma) \cup T_2(\Sigma) \cup \cdots \cup T_n(\Sigma) \subset T(\Sigma \times I), \]
  \[ n \gg 1 \text{ and } T_1 \cong T_2 \cong \cdots \cong T_n \cong T(\Sigma). \]
Hamiltonian evolution

- Hamiltonian triangulation \( T(M) = T_n(\Sigma \times I) \)

WF propagator for a Hamiltonian triangulation

\[
G(L'_s, L_s, t) = Z(T_n(\Sigma \times I)) , \quad \text{where } t = nt_0 .
\]

- **Conjecture I**: \( \Psi(L_s, t) \) satisfies the WdW equation for \( T(\Sigma) \).
- **Conjecture II**: dBB dynamics for \( \Psi(L_s, t) \) is equivalent to the EA dynamics.
Conclusions

- PLQG has finitely many DOF in a compact region, and no infinities.
- The CC spectrum is continuous and depends on 2 free parameters which can be consistently chosen such that the observed CC value is obtained.
- PLQG can be approximated by the GR QFT with a physical cutoff $L_K$ for $N \gg 1$, $M = \Sigma \times I$ and $L_\epsilon \geq L_K \gg l_P$ so that
  \[
  \Gamma(L_\epsilon, \phi_\pi) \approx \Gamma_{qft}(g(x), \phi(x), L_K).
  \]
- One can obtain the running of the masses and the coupling constants with $K \Rightarrow$ PLQG is a microscopic theory whose QFT approximation may, or may not satisfy the asymptotic safety.
- Gravitons are phonons in the $T(M)$ lattice.
- For small $L_\epsilon$ we need a non-perturbative solution of the EA equation. Use minisuperspace models or numerical simulations.
References