Piecewise Flat Quantum Gravity

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Path integral for GR

- M a smooth 4-manifold, g a metric on M then

$$Z(M) = \int \mathcal{D}g \, exp\left(iS_{AH}(g,M)/\hbar\right) = ?$$

- Regge: $T(M)$ a triangulation of M, replace g by $\{L_\epsilon | \, \epsilon \in T_1(M)\}$ and

$$Z(M) = \lim_{T(M)\to M} Z(T(M)) \, .$$

- It is very difficult to find a value of Z in the smooth limit $T(M) \to M$.

Piecewise flat quantum gravity

- Assume that $T(M)$ is the fundamental spacetime structure, i.e. the spacetime is a *piecewise linear* 4-manifold $T(M)$ with a flat metric in each cell (4-symplex). If N is the number of cells of $T(M)$, then for $N \gg 1$, $T(M)$ looks like the smooth manifold M.
- $Z(T(M))$ can be made finite and it is not necessary to define the smooth limit $N \to \infty$. Insted, we need the large-N approximation for the observables. Analogous to the fluid dynamics situation.
- The semiclassical limit will be described by the effective action $\Gamma(L)$, which is computed by using the effective action equation from QFT, in the limit $L_\epsilon \gg l_P = \sqrt{G_N \hbar}$.

Regge state-sum model

- The fundamental DOF are the edge-lengths $L_\epsilon \geq 0$, and

$$Z = \int_{D_E} \prod_{\epsilon=1}^{E} dL_\epsilon \, \mu(L) \, \exp\left(iS_{Rc}(L)/l_P^2 \right) ,$$

  where

$$S_{Rc} = -\sum_{\Delta=1}^{F} A_\Delta(L)\theta_\Delta(L) + \Lambda_c V_4(L) ,$$

  is the Regge action with a cosmological constant. $D_E \subset \mathbf{R}_+^E$ such that the triangle inequalities hold.
- We choose the following PI measure

$$\mu(L) = e^{-V_4(L)/L_0^4} ,$$

  where L_0 is a free parameter.
- We also introduce a classical CC length scale L_c, $\Lambda_c = \pm 1/L_c^2$ $\Rightarrow L_c$ is the second free parameter.

Effective action vs Wilsonian approach

- We will use the *effective action* in order to determine the quantum corrections.
- The EA approach is different from the *Wilsonian* approach to quantization which is used in Quantum Regge Calculus and in Casual Dynamical Triangulations.
- In the Wilsonian approach

$$Z(\kappa,\lambda) = \int_{D_E} \prod_{\epsilon=1}^{E} dL_\epsilon \, \mu(L) \, \exp\left[i\kappa S_R(L)/l_0^2 + i\lambda V_4(L)/l_0^4\right] \, ,$$

  and one looks for the points (κ,λ) where Z'' diverges.
- In the vicinity of a critical point the correlation length diverges $\Leftrightarrow$ transition from the discrete (finitely many DOF) to a continuum (infinitely many DOF) theory.
- The semiclassical limit ($l_P \to 0$) in WA corresponds to the strong-coupling limit $\kappa \to \infty$, hence it can be only studied numerically.

Effective action equation

- Let $\phi : M \to \mathbf{R}$ and $S(\phi) = \int_M d^4x\, \mathcal{L}(\phi, \partial\phi)$ a QFT flat-spacetime action. The effective action $\Gamma(\phi)$ is determined by the integro-differential equation

$$e^{i\Gamma(\phi)/\hbar} = \int \mathcal{D}h \exp\left[\frac{i}{\hbar} S(\phi + h) - \frac{i}{\hbar}\int_M d^4x\, \frac{\delta\Gamma}{\delta\phi(x)} h(x)\right] .$$

- A generic solution $\Gamma(\phi) \in \mathbf{C}$. Wick rotation is used to obtain $\Gamma(\phi) \in \mathbf{R}$: solve the Euclidean-space equation

$$e^{-\Gamma_E(\phi)/\hbar} = \int \mathcal{D}h \exp\left[-\frac{1}{\hbar} S_E(\phi + h) + \frac{1}{\hbar}\int_M d^4x\, \frac{\delta\Gamma_E}{\delta\phi(x)} h(x)\right] ,$$

and then put $x_0 = -it$ in $\Gamma_E(\phi)$.

- Wick rotation is equivalent to $\Gamma(\phi) \to Re\,\Gamma(\phi) + Im\,\Gamma(\phi)$.

Regge effective action

- In the case of a Regge state-sum model

$$e^{i\Gamma(L)/l_P^2} = \int_{D_E(L)} d^E x\, \mu(L+x) e^{iS_{Rc}(L+x)/l_P^2 - i\sum_{\epsilon=1}^{E} \Gamma'_\epsilon(L)x_\epsilon/l_P^2} ,$$

  where $l_P^2 = G_N \hbar$ and $D_E(L)$ is a subset of $\mathbf{R}^E$ obtained by translating D_E by a vector $-L$.
- $D_E(L) \subseteq [-L_1, \infty) \times \cdots \times [-L_E, \infty)$.
- Semiclassical solution

$$\Gamma(L) = S_{Rc}(L) + l_P^2 \Gamma_1(L) + l_P^4 \Gamma_2(L) + \cdots ,$$

  where $L_\epsilon \gg l_P$ and

$$|\Gamma_n(L)| \gg l_P^2 |\Gamma_{n+1}(L)| .$$

Perturbative solution

- Let $L_\epsilon \to \infty$, then $D_E(L) \to \mathbf{R}^E$ and

$$e^{i\Gamma(L)/l_P^2} \approx \int_{\mathbf{R}^E} d^E x\, \mu(L+x) e^{iS_{Rc}(L+x)/l_P^2 - i\sum_{\epsilon=1}^{E} \Gamma'_\epsilon(L) x_\epsilon / l_P^2} .$$

- The reason is $D_E(L) \approx [-L_1, \infty) \times \cdots \times [-L_E, \infty)$ so that

$$\int_{-L}^{\infty} dx\, e^{-zx^2/l_P^2 - wx} = \sqrt{\pi}\, l_P \exp\Big[-\frac{1}{2}\log z + l_P^2 \frac{w^2}{4z}$$

$$+ l_P \frac{e^{-z\bar{L}^2/l_P^2}}{2\sqrt{\pi z}\bar{L}} \left(1 + O(l_P^2/z\bar{L}^2)\right) \Big],$$

where $\bar{L} = L + l_P^2 \frac{w}{2z}$ and $Re\, z = -(\log\mu)''$. The non-analytic terms in $\hbar$ will be absent if

$$\lim_{L\to\infty} e^{-z\bar{L}^2/l_P^2} = 0 \Leftrightarrow (\log\mu)'' < 0 \; for \; L \to \infty .$$

Hence the perturbative solution exists for the exponentially damped measures.

Perturbative solution

- For $D_E(L) = \mathbf{R}^E$ and $\mu(L) = const.$ the perturbative solution is given by the EA diagrams

$$\Gamma_1 = \frac{i}{2} Tr \log S''_{Rc} \,, \quad \Gamma_2 = \langle S_3^2 G^3 \rangle + \langle S_4 G^2 \rangle \,,$$

$$\Gamma_3 = \langle S_3^4 G^6 \rangle + \langle S_3^2 S_4 G^5 \rangle + \langle S_3 S_5 G^4 \rangle + \langle S_4^2 G^4 \rangle + \langle S_6 G^3 \rangle \,, \ldots$$

  where $G = i(S''_{Rc})^{-1}$ is the propagator and $S_n = iS^{(n)}_{Rc}/n!$ for $n > 2$, are the vertex weights.
- When $\mu(L) \neq const.$, the perturbative solution is given by

$$\Gamma(L) = \bar{S}_{Rc}(L) + l_P^2 \bar{\Gamma}_1(L) + l_P^4 \bar{\Gamma}_2(L) + \cdots \,,$$

  where

$$\bar{S}_{Rc} = S_{Rc} - i l_P^2 \log \mu \,,$$

  while $\bar{\Gamma}_n$ is given by the sum of n-loop EA diagrams with $\bar{G}$ propagators and $\bar{S}_n$ vertex weights.

Perturbative solution

- Therefore

$$\Gamma_1 = -i \log \mu + \frac{i}{2} Tr \log S''_{Rc}$$

$$\Gamma_2 = \langle S_3^2 G^3 \rangle + \langle S_4 G^2 \rangle + Res[l_P^{-4} Tr \log \bar{G}] \,,$$

$$\Gamma_3 = \langle S_3^4 G^6 \rangle + \cdots + \langle S_6 G^3 \rangle + Res[l_P^{-6} Tr \log \bar{G}]$$

$$+ Res[l_P^{-6} \langle \bar{S}_3^2 \bar{G}^3 \rangle] + Res[l_P^{-6} \langle \bar{S}_4 \bar{G}^2 \rangle] \,, \cdots$$

- Since $\log \mu(L) = O\left((L/L_0)^4\right)$ and for

$$L_\epsilon > L_c \,, \quad L_0 > \sqrt{l_P L_c}$$

we get the following large-L asymptotics

$$\Gamma_1(L) = O(L^4/L_0^4) + \log O(L^2/L_c^2) + \log \theta(L) + O(L_c^2/L^2)$$

and

$$\Gamma_{n+1}(L) = O\left((L_c^2/L^4)^n\right) + L_{0c}^{-2n} O\left((L_c^2/L^2)\right) \,,$$

where $L_{0c} = L_0^2/L_c$.

QG cosmological constant

- For $L_\epsilon \gg l_P$ and $L_0 \gg \sqrt{l_P\, L_c}$ the series

$$\sum_{n\geq 0} (l_P)^{2n} \Gamma_n(L)$$

is semiclassical.

- Let $\Gamma \to \Gamma / G_N$ so that $S_{eff} = (Re\,\Gamma \pm Im\,\Gamma)/G_N$
- One-loop CC

$$S_{eff} = \frac{S_{Rc}}{G_N} \pm \frac{l_P^2}{G_N L_0^4} V_4 \pm \frac{l_P^2}{2G_N} Tr \log S''_{Rc} + O(l_P^4) \Rightarrow$$

$$\Lambda = \Lambda_c \pm \frac{l_P^2}{2L_0^4} = \Lambda_c + \Lambda_{qg} \,.$$

- The one-loop cosmological constant is exact because there are no $O(L^4)$ terms beyond the one-loop order.

QG cosmological constant

- This is a consequence of the large-L asymptotics

$$\log \bar{S}''_{Rc}(L) = \log O(L^2/\bar{L}_c^2) + \log\theta(L) + O(\bar{L}_c^2/L^2)$$

$$\bar{\Gamma}_{n+1}(L) = O\left((\bar{L}_c^2/L^4)^n\right) \,,$$

where $\bar{L}_c^2 = L_c^2\left[1 + i l_P^2(L_c^2/L_0^4)\right]^{-1/2}$.

- Note that $L_c \geq l_P$ implies $L_0 \gg l_P$, so that

$$l_P^2|\Lambda_{qg}| = \frac{1}{2}\left(\frac{l_P}{L_0}\right)^4 \ll 1\,.$$

If $L_c < l_P$ then $L_0 \gg l_P$ is consistent with the semiclassical approximation.

- If $\Lambda_c = 0$, the observed value of Λ is obtained for $L_0 \approx 10^{-5} m$ so that $l_P^2\Lambda \approx 10^{-122}$.

Smooth-spacetime limit

- Smooth spacetime is described by $T(M)$ with $E \gg 1 \Rightarrow$

$$S_R(L) \approx \frac{1}{2}\int_M d^4x\sqrt{|g|}\,R(g)\,,$$

$$\Lambda V_4(L) \approx \Lambda \int_M d^4x\sqrt{|g|} = \Lambda\, V_M\,,$$

- For $L_\epsilon \geq L_K \gg l_P$ and $E \gg 1$

$$Tr\log S_R''(L) \approx \int_M d^4x\sqrt{|g|}\left[a(L_K)R^2 + b(L_K)R_{\mu\nu}R^{\mu\nu}\right]\,.$$

- When $M = \Sigma \times I$, L_K defines a QFT momentum UV cutoff $\hbar/L_K$. LHC experiments $\Rightarrow L_K < 10^{-20}\mathrm{m} \Leftrightarrow \hbar K > 10$ TeV.

Coupling of matter

- Scalar field on M

$$S_m(g,\phi) = \frac{1}{2}\int_M d^4x\sqrt{|g|}\,[g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - U(\phi)]\ ,$$

where $U = \frac{1}{2}\omega^2\phi^2 + \lambda\,\phi^4$.

- On $T(M)$ we get

$$S_m = \frac{1}{2}\sum_\sigma V_\sigma(L)\sum_{k,l} g_\sigma^{kl}(L)\,\phi'_k\,\phi'_l - \frac{1}{2}\sum_\pi V^*_\pi(L)\,U(\phi_\pi)\,,$$

where $\phi'_k = (\phi_k - \phi_0)/L_{0k}$.

- The total classical action

$$S(L,\phi) = \frac{1}{G_N}S_{Rc}(L) + S_m(L,\phi)\,.$$

Coupling of matter

- The EA equation

$$e^{i\Gamma(L,\phi)/l_P^2} = \int_{D_E(L)} d^E x \int_{\mathbf{R}^V} d^V\chi \exp\left[i\bar{S}_{Rm}(L+x,\phi+\chi)/l_P^2 \right.$$

$$\left. -i\sum_\epsilon \frac{\partial\Gamma}{\partial L_\epsilon} x_\epsilon/l_P^2 - i\sum_\pi \frac{\partial\Gamma}{\partial\phi_\pi}\chi_\pi/l_P^2 \right],$$

  where $\bar{S}_{Rm} = \bar{S}_{Rc} + G_N S_m(L,\phi)$.

- Perturbative solution

$$\Gamma(L,\phi) = S(L,\phi) + l_P^2\Gamma_1(L,\phi) + l_P^4\Gamma_2(L,\phi) + \cdots$$

  is semiclassical for $L_\epsilon \gg l_P$, $L_0 \gg l_P$ and $|\sqrt{G_N}\,\phi| \ll 1$. This can be checked in $E = 1$ toy model

$$S(L,\phi) = (L^2 + L^4/L_c^2)\theta(L) + L^2\theta(L)\phi^2(1 + \omega^2 L^2 + \lambda\phi^2 L^2)\,.$$

Coupling of matter

- $\Gamma(L,\phi) = \Gamma_g(L) + \Gamma_m(L,\phi)$
- $\Gamma_m(L,\phi) = V_4(L)\, U_{eff}(\phi)$ for constant ϕ and $U_{eff}(0) = 0$.
- $\Gamma_g(L) = \Gamma_{pg}(L) + \Gamma_{mg}(L)$ and

$$\Gamma_{mg}(L) \approx \Lambda_m V_M + \Omega_m(R,K)$$

  in the smooth-manifold approximation and $K = 1/L_K$.
- $\Omega_m = \Omega_1 l_P^2 + O(l_P^4)$ and

$$\Omega_1(R,K) = a_1 K^2 \int_M d^4x \sqrt{|g|}\, R +$$

$$\log(K/\omega) \int_M d^4x \sqrt{|g|} \left[a_2 R^2 + a_3 R^{\mu\nu} R_{\mu\nu} + a_4 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + a_5 \nabla^2 R \right]$$

$$+ O(L_K^2/L^2)\,.$$

CC spectrum

- The effective CC

$$\Lambda = \Lambda_c + \Lambda_{qg} + \Lambda_m \,,$$

where

$$\Lambda_m \approx \sum_\gamma v(\gamma, K)$$

and $v(\gamma, K)$ is a 1PI vacuum FD for S_m with the cutoff K.

- For $K \gg \omega$

$$\sum_\gamma v(\gamma, K) \approx l_P^2\, K^4 \Big[c_1 \ln(K^2/\omega^2) + \sum_{n \geq 2} c_n(\bar\lambda)^{n-1} (\ln(K^2/\omega^2))^{n-2}$$

$$+ \sum_{n\geq 4} d_n (\bar\lambda)^{n-1} (K^2/\omega^2)^{n-3} \Big] \,,$$

where $\bar\lambda = l_P^2 \lambda$.

- In QFT $\Lambda_m = \lim_{K\to\infty} \sum_\gamma v(\gamma, K) = \infty$. Not clear how to define a NP value.

CC spectrum

- In PLQG the exact solution of the EA equation will give a finite and cutoff-independent value

$$\Lambda_m = V(\omega^2, \lambda, l_P^2)\,.$$

- Hence

$$\Lambda = \pm\frac{1}{L_c^2} + \frac{l_P^2}{2L_0^4} + V(\omega^2, \lambda, l_P^2)\,.$$

- We can fix the free parameter L_c by setting

$$\Lambda_c + \Lambda_m = 0\,,$$

while L_0 can be fixed by matching Λ to the observed CC value:

$$\Lambda = l_P^2/2L_0^4 \Rightarrow L_0 \approx 10^{-5}\, m\,.$$

- This value is consistent since $L_0 \gg l_P$.

CC problem in quantum gravity

- The CC problem in QG has two parts (Polchinski 2006):
  (1) Show that the observed CC value is in the CC spectrum.
  (2) Explain why CC takes the observed value.
- PLQG solves (1).
- String theory may solve (1): CC spectrum is discrete with 10^{500} values (Russo and Polchinski 2000) and positive values are allowed (KKLT 2003) $\Rightarrow$ plausible to assume that the CC spectrum is sufficiently dense around zero.
- (2) is a much harder question, and one "explanation" is the antrophic principle.

Quantum cosmology

- The EA formalism is only applicable for $M = \Sigma \times I$.
- In the case

$$M \neq \Sigma \times I\,, \quad \partial M = \Sigma\,,$$

one can define a Hartle-Hawking wf of the Universe

$$\Psi_0(L_s) = Z\,(T(M)) \;\; with \;\; L|_{T(\Sigma)} = L_s\,.$$

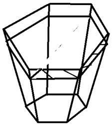
- Big Bang $\Leftrightarrow M = M_0 \cup (\Sigma \times I)$.
- Cosmological bounce $\Leftrightarrow M = \Sigma \times \mathbf{R}$.
- Hamiltonian evolution on $T(\Sigma \times I)$ possible if

$$T_1(\Sigma) \cup T_2(\Sigma) \cup \cdots \cup T_n(\Sigma) \subset T(\Sigma \times I)\,,$$

$n \gg 1$ and $T_1 \cong T_2 \cong \cdots \cong T_n \cong T(\Sigma)$.

Hamiltonian evolution

- Hamiltonian triangulation $T(M) = T_n(\Sigma \times I)$



- WF propagator for a Hamiltonian triangulation

$$G(L'_s, L_s, t) = Z(T_n(\Sigma \times I)), \quad where\, t = nt_0 .$$

- **Conjecture I**: $\Psi(L_s, t)$ satisfies the WdW equation for $T(\Sigma)$.
- **Conjecture II**: dBB dynamics for $\Psi(L_s, t)$ is equivalent to the EA dynamics.

Conclusions

- PLQG has finitely many DOF in a compact region, and no infinities.
- The CC spectrum is continious and depends on 2 free parameters which can be consistently choosen such that the observed CC value is obtained.
- PLQG can be approximated by the GR QFT with a physical cutoff L_K for $N \gg 1$, $M = \Sigma \times I$ and $L_\epsilon \geq L_K \gg l_P$ so that

$$\Gamma(L_\epsilon, \phi_\pi) \approx \Gamma_{qft}(g(x), \phi(x), L_K) \,.$$

- One can obtain the running of the masses and the coupling constants with $K \Rightarrow$ PLQG is a microscopic theory whose QFT approximation may, or may not satisfy the asymptotic safety.
- Gravitons are phonons in the $T(M)$ lattice.
- For small L_ϵ we need a non-perturbative solution of the EA equation. Use minisuperspace models or numerical simulations.

References

- A. Miković, *Effective actions for Regge state-sum models of quantum gravity*, Adv. Theor. Math. Phys. 21 (2017) 631
- A. Miković and M. Vojinović, *Solution to the Cosmological Constant Problem in a Regge Quantum Gravity Model*, Europhys. Lett. 110, 40008 (2015)
- A. Miković and M. Vojinović, *Cosmological constant in a PLQG theory*, arxiv:1407.1124.
- A. Miković, *Spin-cube models of quantum gravity*, Rev. Math. Phys. 25 (2013) 10, 1343008.
- A. Miković and M. Vojinović, *Effective action and semiclassical limit of spin-foam models*, Class. Quant. Grav. 28 (2011) 225004.
- A. Miković and M. Vojinović, *Effective action for EPRL/FK spin-foam models*, J. Phys.: Conf. Ser. 360, (2012) 012049.