T-dualization of type II pure spinor superstring in double space

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Outline of the talk

Superstrings

- There are five consistent superstring theories. They are connected by web of T and S dualities.
- There are three approaches to superstring theory: NSR (Neveu-Schwarz-Ramond), GS (Green-Schwarz) and pure spinor formalism (N. Berkovits, hep-th/0001035).
- T-duality transformation does not change the physical content of the theory.
- Well known bosonic and recently discovered fermionic T-duality.

Idea od double space

- Double space $=$ initial coordinates plus T-dual partners - Siegel, Duff, Tseytlin about 25 years ago.
- Interest for this subject emerged again (Hull, Berman, Zwiebach) in the context of T-duality as $O(d, d)$ transformation.
- The approach of Duff has been recently improved when the T-dualization along some subset of the initial and corresponding subset of the T-dual coordinates has been interpreted as permutation of these subsets in the double space coordinates (arXiv:1505.06044, 1503.05580). All calculations are made in full double space.
- In double space T-duality is a symmetry transformation.

General pure spinor action for type II superstring

- We start from the general pure spinor action for type II superstring (arXiv: 0405072)

$$
\begin{aligned}
S = \int d^2\xi \Big[& \partial_+\theta^\alpha A_{\alpha\beta}\partial_-\bar{\theta}^\beta + \partial_+\theta^\alpha A_{\alpha\mu}\Pi_-^\mu + \Pi_+^\mu A_{\mu\alpha}\partial_-\bar{\theta}^\alpha \\
+ & \Pi_+^\mu A_{\mu\nu}\Pi_-^\nu + d_\alpha E^\alpha{}_\beta \partial_-\bar{\theta}^\beta + d_\alpha E^\alpha{}_\mu \Pi_-^\mu + \partial_+\theta^\alpha E_\alpha{}^\beta \bar{d}_\beta + \Pi_+^\mu E_\mu{}^\beta \bar{d}_\beta \\
+ & d_\alpha \mathrm{P}^{\alpha\beta}\bar{d}_\beta + \frac{1}{2}N_+^{\mu\nu}\Omega_{\mu\nu,\beta}\partial_-\bar{\theta}^\beta + \frac{1}{2}N_+^{\mu\nu}\Omega_{\mu\nu,\rho}\Pi_-^\rho + \frac{1}{2}\partial_+\theta^\alpha\Omega_{\alpha,\mu\nu}\bar{N}_-^{\mu\nu} \\
+ & \frac{1}{2}\Pi_+^\mu\Omega_{\mu,\nu\rho}\bar{N}_-^{\nu\rho} + \frac{1}{2}N_+^{\mu\nu}\bar{C}_{\mu\nu}{}^\beta\bar{d}_\beta + \frac{1}{2}d_\alpha C^\alpha{}_{\mu\nu}\bar{N}_-^{\mu\nu} \\
+ & \frac{1}{4}N_+^{\mu\nu}S_{\mu\nu,\rho\sigma}\bar{N}_-^{\rho\sigma}\Big] + S_\lambda + S_{\bar{\lambda}} \,.
\end{aligned}
\tag{1}
$$

Bosonic T-duality - assumptions and approximations

- Bosonic T-dualization - we assume that background fields are independent of x^μ. In mentioned reference, expressions for background fields as well as action are obtained in an iterative procedure as an expansion in powers of θ^α and $\bar{\theta}^\alpha$. Every step in iterative procedure depends on previuous one, so, for mathematical simplicity, we consider only basic (θ and $\bar{\theta}$ independent) components.

Fermionic T-duality - assumptions and consistency check

- Fermionic T-dualization - we assume that θ^α and $\bar{\theta}^\alpha$ are Killing directions. Consequently, auxiliary superfirlds are zero according to arXiv: 0405072. If we assume that rest of background fields are constant then their curvatures are zero. Using space-time field equations we confirmed the consistency of the choice of constant $\mathrm{P}^{\alpha\beta}$.

Action

- In both cases, under introduced assumptions, action gets the form

$$S = \kappa \int_\Sigma d^2\xi \left[\partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu + \frac{1}{4\pi\kappa} \Phi R^{(2)}\right] + \int_\Sigma d^2\xi \left[-\pi_\alpha \partial_-(\theta^\alpha + \Psi^\alpha_\mu x^\mu) + \partial_+(\bar\theta^\alpha + \bar\Psi^\alpha_\mu x^\mu)\bar\pi_\alpha + \frac{e^{\frac{\Phi}{2}}}{2\kappa} \pi_\alpha F^{\alpha\beta} \bar\pi_\beta\right] \quad (2$$

- Definitions: $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$, Φ is dilaton field, Ψ^α_μ and $\bar\Psi^\alpha_\mu$ are NS-R fields and $F^{\alpha\beta}$ is R-R field strength. Momenta π_α and $\bar\pi_\alpha$ are canonically conjugated to θ^α and $\bar\theta^\alpha$. All spinors are Majorana-Weyl ones.
- All background fields are constant.

Busher bosonic T-duality

- Global shift symmetry exists $x^a \to x^a + b$, where index a is subset of μ.
- We introduce gauge fields $v_\pm^a$ and make change in the action $\partial_\pm x^a \to \partial_\pm x^a + v_\pm^a$.
- Additional term in the action

$$S_{gauge}(y, v_\pm) = \frac{1}{2}\kappa \int_\Sigma d^2\xi \left(v_+^a \partial_- y_a - \partial_+ y_a v_-^a \right) \ ,$$

  where y_a is Lagrange multiplier. It makes $v_\pm^a$ to be unphysical degrees of freedom.
- On the equations of motion for y_a we get initial action, while, fixing x^a to zero, on th equations of motion for $v_\pm^a$ we get T-dual action.

Transformation laws

- Solution of the equation of motion for y_a is $v^a_\pm = \partial_\pm x^a$. Combining this solution with equations of motion for gauge fields $v^a_\pm$ we obtain T-dual transformation laws

$$\partial_\pm x^a \cong -2\kappa\hat{\theta}^{ab}_\pm \Pi_{\mp bi}\partial_\pm x^i - \kappa\hat{\theta}^{ab}_\pm(\partial_\pm y_b - J_{\pm b})\,, \tag{3}$$

$$\partial_\pm y_a \cong -2\Pi_{\mp ab}\partial_\pm x^b - 2\Pi_{\mp ai}\partial_\pm x^i + J_{\pm a}\,. \tag{4}$$

Here $J_{\pm\mu} = \pm\frac{2}{\kappa}\Psi^\alpha_{\pm\mu}\pi_{\pm\alpha}$ and $\theta^{ac}_\pm\Pi_{\mp cb} = \frac{1}{2\kappa}\delta^a{}_b$, where

$$\Psi^\alpha_{+\mu} \equiv \Psi^\alpha_\mu\,, \quad \Psi^\alpha_{-\mu} \equiv \bar{\Psi}^\alpha_\mu\,, \quad \pi_{+\alpha} \equiv \pi_\alpha\,, \quad \pi_{-\alpha} \equiv \bar{\pi}_\alpha\,. \tag{5}$$

Transformation laws in double space

- In double space spanned by $Z^M = (x^\mu, y_\mu)^T$ they are of the form

$$\partial_\pm Z^M \cong \pm \Omega^{MN} \left(\mathcal{H}_{NP} \partial_\pm Z^P + J_{\pm N} \right) , \tag{6}$$

where

$$\mathcal{H}_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2\, B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}\, B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix} , \tag{7}$$

is so called generalized metric, while

$$\Omega^{MN} = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix} , J_{\pm M} = \begin{pmatrix} 2(\Pi_\pm G^{-1})_\mu{}^\nu J_{\pm\nu} \\ -(G^{-1})^{\mu\nu} J_{\pm\nu} \end{pmatrix} . \tag{8}$$

Ω^{MN} is constant symmetric matrix and it is known as $SO(D,D)$ invariant metric. Here $G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$.

T-duality as permutation in double space

- T-dualization in double space is represented by permutation

$$
{}_a Z^M \equiv \begin{pmatrix} y_a \\ x^i \\ x^a \\ y_i \end{pmatrix} = (\mathcal{T}^a)^M{}_N Z^N \equiv \begin{pmatrix} 0 & 0 & 1_a & 0 \\ 0 & 1_i & 0 & 0 \\ 1_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_i \end{pmatrix} \begin{pmatrix} x^a \\ x^i \\ y_a \\ y_i \end{pmatrix} .
$$

T-duality as permutation in double space

- Demanding that ${}_aZ^M$ has the transformation law of the same form as initial coordinates Z^M, we find the T-dual generalized metric

$$ {}_a\mathcal{H}_{MN} = (\mathcal{T}^a)_M{}^K \mathcal{H}_{KL} (\mathcal{T}^a)^L{}_N \,, \tag{9} $$

and T-dual current

$$ {}_aJ_{\pm M} = (\mathcal{T}^a)_M{}^N J_{\pm N} \,. \tag{10} $$

NS-NS background fields

- From (9) we obtain the T-dual NS-NS background fields which are in full agreement with those obtained by Buscher procedure

$$
{}_a\Pi_{\pm}^{ab} = \frac{\kappa}{2}\hat{\theta}_{\mp}^{ab}\,, \qquad {}_a\Pi^{a}_{\pm\, i} = \kappa\hat{\theta}_{\mp}^{ab}\Pi_{\pm bi}\,,
$$

$$
{}_a\Pi_{\pm i}{}^{a} = -\kappa\Pi_{\pm ib}\hat{\theta}_{\mp}^{ba}\,, \qquad {}_a\Pi_{\pm ij} = \Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\hat{\theta}_{\mp}^{ab}\Pi_{\pm bj}\,.
$$

NS-R background fields

- From (10) we obtain the form of the T-dual NS-R fields

$$ {}_a\Psi^{\alpha a} = \kappa \hat\theta_+^{ab} \Psi_b^\alpha \,, \quad {}_a\bar\Psi^{\alpha a} = \kappa \, {}_a\Omega^\alpha{}_\beta \hat\theta_-^{ab} \bar\Psi_b^\beta \,. \qquad (11) $$

$$ {}_a\Psi_i^\alpha = \Psi_i^\alpha - 2\kappa \Pi_{-ib} \hat\theta_+^{ba} \Psi_a^\alpha \,, \quad {}_a\bar\Psi_i^\alpha = {}_a\Omega^\alpha{}_\beta (\bar\Psi_i^\beta - 2\kappa \Pi_{+ib} \hat\theta_-^{ba} \bar\Psi_a^\beta) \,. \qquad (12) $$

- From transformation laws we see that two chiral sectors transform differently. Consequently, there are two sets of vielbeins in T-dual picture as well two sets of gamma matrices. This T-dual vielbeins are connected by Lorentz transformation, while spinorial representation of this Lorentz transformation, ${}_a\Omega^\alpha{}_\beta$, relates two sets of gamma matrices. In order to have unique set of gamma matrices, we have to multiply one fermionic index by ${}_a\Omega^\alpha{}_\beta$.

R-R field strength

- R-R field strength couples fermionic momenta and, consequently, its T-dual can not be read from transformation law.
- From the demand that term in the action is T-dual invariant, we obtain the form of the T-dual R-R field stregth

$$e^{\frac{^a\Phi}{2}}\, {}_aF^{\alpha\beta} = (e^{\frac{\Phi}{2}} F^{\alpha\gamma} + c\Psi^\alpha_a \hat{\theta}^{ab}_- \bar{\Psi}^\gamma_b)_a \Omega_\gamma{}^\beta \,, \tag{13}$$

where c is an arbitrary constant. For the specific value of c, we get the same expression as in Buscher procedure.

Basic facts

- In last years it was seen that tree level superstring theories on certain supersymmetric backgrounds admit a symmetry which is called fermionic T-duality.
- This is a redefinition of the fermionic worldsheet fields similar to the redefinition we perform on bosonic variables when we do an ordinary T-duality.
- Technically, the procedure is the same as in the bosonic case up to the fact that dualization will be done along θ^α and $\bar{\theta}^\alpha$ directions.

Action

- On the equations of motion for π_α and $\bar{\pi}_\alpha$ action (2) becomes

$$
\begin{aligned}
S = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \left[\Pi_{+\mu\nu} + \frac{1}{2} \bar{\Psi}^\alpha_\mu (P^{-1})_{\alpha\beta} \Psi^\beta_\nu \right] \partial_- x^\nu \\
+ \quad & \frac{1}{4\pi} \int_\Sigma d^2\xi \Phi R^{(2)} \\
+ \quad & \frac{\kappa}{2} \int_\Sigma d^2\xi \left[\partial_+ \bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \partial_- \theta^\beta + \partial_+ \bar{\theta}^\alpha (P^{-1}\Psi)_{\alpha\mu} \partial_- x^\nu \right. \\
+ \quad & \left. \partial_+ x^\mu (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_- \theta^\alpha \right] .
\end{aligned}
$$

Fixing the chiral gauge invariance

- In the above action θ^α appears only in the form $\partial_- \theta^\alpha$ and $\bar\theta^\alpha$ in the form $\partial_+ \bar\theta^\alpha$.
- Using the BRST formalism we fix theis chiral gauge invariance adding to the action

$$S_{gf} = -\frac{\kappa}{2} \int d^2\xi \partial_- \bar\theta^\alpha (\alpha^{-1})_{\alpha\beta} \partial_+ \theta^\beta \,, \tag{14}$$

where $\alpha^{\alpha\beta}$ is arbitrary non singular matrix.

Transformation laws

- Applying the same mathematical procedure as in the case of the bosonic T-dualization, we have

$$\partial_-\theta^\alpha \cong -P^{\alpha\beta}\partial_-\vartheta_\beta - \Psi_\mu^\alpha \partial_- x^\mu \,, \partial_+\bar\theta^\alpha \cong \partial_+\bar\vartheta_\beta P^{\beta\alpha} - \partial_+ x^\mu \bar\Psi_\mu^\alpha \,, \tag{15}$$

$$\partial_+\theta^\alpha \cong -\alpha^{\alpha\beta}\partial_+\vartheta_\beta \,, \partial_-\bar\theta^\alpha \cong \partial_-\bar\vartheta_\beta \alpha^{\beta\alpha} \,, \tag{16}$$

where ϑ_α and $\bar\vartheta_\alpha$ are T-dual fermionic coordinates.

Transformation laws in double space

- Let us introduce double fermionic coordinates

$$\Theta^A = \begin{pmatrix} \theta^\alpha \\ \vartheta_\alpha \end{pmatrix}, \quad \bar\Theta^A = \begin{pmatrix} \bar\theta^\alpha \\ \bar\vartheta_\alpha \end{pmatrix}. \tag{17}$$

- Transformation laws in double space are of the form

$$\partial_-\Theta^A \cong -\Omega^{AB}\left[\mathcal{F}_{BC}\partial_-\Theta^C + \mathcal{J}_{-B}\right],$$

$$\partial_+\bar\Theta^A \cong \left[\partial_+\bar\Theta^C\mathcal{F}_{CB} + \bar{\mathcal{J}}_{+B}\right]\Omega^{BA},$$

$$\partial_+\Theta^A \cong -\Omega^{AB}\mathcal{A}_{BC}\partial_+\Theta^C\,, \partial_-\bar\Theta^A \cong \partial_-\bar\Theta^C\mathcal{A}_{CB}\Omega^{BA}.$$

Generalized metric and currents

- The generalized metric and the matrix $\mathcal{A}_{AB}$ are

$$\mathcal{F}_{AB} = \begin{pmatrix} (P^{-1})_{\alpha\beta} & 0 \\ 0 & P^{\gamma\delta} \end{pmatrix}, \mathcal{A}_{AB} = \begin{pmatrix} (\alpha^{-1})_{\alpha\beta} & 0 \\ 0 & \alpha^{\gamma\delta} \end{pmatrix}.$$

- The currents are of the form

$$\bar{\mathcal{J}}_{+A} = \begin{pmatrix} (\bar{\Psi} P^{-1})_{\mu\alpha}\partial_+ x^\mu \\ -\bar{\Psi}^\alpha_\mu \partial_+ x^\mu \end{pmatrix}, \mathcal{J}_{-A} = \begin{pmatrix} (P^{-1}\Psi)_{\alpha\mu}\partial_- x^\mu \\ \Psi^\alpha_\mu \partial_- x^\mu \end{pmatrix}.$$

Fermionic T-dualization as permutation

- T-dual coordinates are

$$^\star\Theta^A = \mathcal{T}^A{}_B \Theta^B\,, \quad ^\star\bar{\Theta}^A = \mathcal{T}^A{}_B \bar{\Theta}^B\,,$$

where

$$\mathcal{T}^A{}_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,$$

is permutation matrix.

Fermionic T-dualization as permutation

- Demanding that T-dual coordinates transformation laws are of the same form as those for initial coordinates we get

$$^\star\mathcal{F}_{AB} = \mathcal{T}_A{}^C \mathcal{F}_{CD} \mathcal{T}^D{}_B\,, \quad ^\star\bar{\mathcal{J}}_{+A} = \mathcal{T}_A{}^B \bar{\mathcal{J}}_{+B}\,, \quad ^\star\mathcal{J}_{-A} = \mathcal{T}_A{}^B \mathcal{J}_{-B}\,.$$

- The matrix $\mathcal{A}_{AB}$ transforms as

$$^\star\mathcal{A}_{AB} = \mathcal{T}_A{}^C \mathcal{A}_{CD} \mathcal{T}^D{}_B = (\mathcal{A}^{-1})_{AB}\,. \tag{18}$$

Background fields

- From these relations we obtain the R-R and NS-R T-dual background fields in the same form as in the Buscher procedure

$$^\star P_{\alpha\beta} = (P^{-1})_{\alpha\beta}\,, \quad (^\star\alpha)_{\alpha\beta} = (\alpha^{-1})_{\alpha\beta}\,,$$

$$^\star\Psi_{\alpha\mu} = (P^{-1})_{\alpha\beta}\Psi^\beta_\mu\,, \quad ^\star\bar\Psi_{\alpha\mu} = -\bar\Psi^\beta_\mu(P^{-1})_{\beta\alpha}\,.$$

NS-NS background fields

- $\Pi_{+\mu\nu}$ is coupled by x's and we can not read the T-dual field from transformation laws.
- As in the case of bosonic T-dualization, assuming that this term is invariant under T-dualization, we get the appropriate fermionic T-dual

$$^\star\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + c\bar{\Psi}^\alpha_\mu (P^{-1})_{\alpha\beta} \Psi^\beta_\nu \,, \tag{19}$$

where c is an arbitrary constant.

Concluding remarks

- We represented both kind of T-dualizations of type II superstring as permutation symmetry in double space.
- The successive T-dualizations make a group called T-duality group. In the case of type II superstring fermionic T-duality transformations are performed by the same matrices $\mathcal{T}^a$ as in the bosonic string case. Consequently, the corresponding T-duality group is the same.
- In the bosonic case there is an advantage of this approach. In one equation all T-dual theories (for any subset x^a) are contained. We do not have to repeat procedure for each specific choice of x^a. This kind of approach could be helpful in better understanding of M-theory.