Noncommutative $SO(2,3)$ model

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Overview

Introduction

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GR and QFT are cornerstones of modern physics, but both theories suffer from singularities. QFT and GR encounter problems small distances. There is no consistent (i. e. renormalizable and unitary) quantum theory of gravity. Modifications of QFT and GR are needed; point-particles or/and space-time structure.
Many attempts: String theory, Loop Quantum Gravity,..
One possibility is noncommutativity among space time coordinates. It is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x) \ .$$

Canonical noncommutativity

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} = \mathrm{const.}$$

Different models are constructed on canonical NC spacetime: ϕ^4,QED, standard model, SUSY models;
renormalizability, unitarity, phenomenological consequences, ...

Noncommutative Gravity

GR is based on diffeomorphism symmetry. It is difficult to generalize this symmetry to NC space-time.

Many attempts:

• Twist approach: Commutative diffeomorphisms are replaced by twisted diffeomorphisms

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• Frame formalism, operator description [Burić, Madore '14; Fritz, Majid '16]

• Sieberg-Witten approach

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2 M. A. Cardella and D. Zanon, CQG **20** (2003) L95

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Commutative model

Consider a gauge theory with $SO(2,3)$ as a gauge group.
$SO(2,3)$ is the isometry group anti de Sitter space.
Anti de Sitter space is a maximally symmetric space with a negative constant curvature.
M_{AB}-generators of $SO(2,3)$ group
$A, B, \dots$ take values $0, 1, 2, 3, 5$.
Commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD} M_{BC} + \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC}), \quad (1)$$

$\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$ is $5D$ metric.

Clifford generators Γ_A in $5D$ Minkowski space satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \ . \tag{2}$$

M_{AB} are

$$M_{AB} = \frac{i}{2}[\Gamma_A, \Gamma_B] \ . \tag{3}$$

$\gamma_a, \ (a = 0, 1, 2, 3)$ are the gamma matrices in $4D$ Minkovski spacetime
The gamma matrices in $5D$ are

$$\Gamma_A = (i\gamma_a\gamma_5, \gamma_5) \ .$$

γ_5 is defined by $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

It is easy to show that

$$\begin{aligned} M_{ab} &= \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab} \ , \\ M_{5a} &= \frac{i}{2}\gamma_a \ . \end{aligned} \tag{4}$$

If we introduce momenta $P_a = \frac{1}{l} M_{a5}$,where l is a constant with dimensions of length AdS algebra (1) becomes

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}) \\ [M_{ab}, P_c] &= i(\eta_{bc} P_a - \eta_{ac} P_b) \\ [P_a, P_b] &= -i \frac{1}{l^2} M_{ab} \ . \end{aligned} \tag{5}$$

In the limit $l \to \infty$ AdS algebra reduces usual Poincare algebra in $4D$ spacetime. (Wigner-Inonu contraction)

$SO(2,3)$ gauge potential:

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma^{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a \qquad (6)$$

Transformation law

$$\delta_\epsilon \omega_\mu = \partial_\mu \epsilon + i[\epsilon, \omega_\mu] \qquad (7)$$

Decomposition: ω_μ^{AB} *to* ω_μ^{ab}, ω_μ^{a5},
ω_μ^{ab} is a spin connection
$\omega_\mu^{a5} = \frac{1}{l} e_\mu^a$ are vierbeins (tetrads).

The field strength

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2} F^{AB}_{\mu\nu} M_{AB} \\ &= \frac{1}{2} F^{ab}_{\mu\nu} M_{ab} + F^{a5}_{\mu\nu} M_{a5} \ , \end{aligned} \tag{8}$$

where

$$F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} - \frac{1}{l^2}(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu) \ . \tag{9}$$

Reiman curvature tensor is

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{cb}_\nu - \omega^{bc}_\mu \omega^{ca}_\nu \tag{10}$$

Torsion

$$l F^{a5}_{\mu\nu} = D_\mu e^a_\nu - D_\nu e^a_\mu = T^a_{\mu\nu} \tag{11}$$

Under the local anti de Sitter transformation field strength transforms as

$$\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] \tag{12}$$

We introduce an auxiliary field $\phi = \phi^A \Gamma_A$.
Transformation law:

$$\delta_\epsilon \phi = i[\epsilon, \phi] \tag{13}$$

This field satisfies a constraint $\phi_A \phi^A = l^2$.
Action:

$$\begin{aligned} S_1 &= \frac{il}{64\pi G_N} \int \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left(F_{\mu\nu} F_{\rho\sigma} \phi\right) \\ S_2 &= \frac{1}{128\pi G_N l} \mathrm{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} D_\rho \phi D_\sigma \phi \phi + c.c, \qquad (14) \\ S_3 &= -\frac{i}{128\pi G_N l} \mathrm{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi \qquad (15) \end{aligned}$$

S_1 is Stelle-West action (1980).

$$S = c_1 S_1 + c_2 S_2 + c_3 S_3 \tag{16}$$

is invariant under the $SO(2,3)$ gauge transformations.
We reduce the local anti de Sitter symmetry down to the local Lorentz symmetry:

$$SO(2,3) \to SO(1,3)$$

After symmetry breaking (i. e. $\phi^a = 0, \phi^5 = l$) these actions reduce to

$$\begin{aligned}
S_1 &= -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big(\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ ab}R_{\rho\sigma}^{\ cd} + \sqrt{-g}\big(R - \frac{6}{l^2}\big)\Big)\,, \\
S_2 &= -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big(R - \frac{12}{l^2}\Big), \\
S_3 &= -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big(-\frac{12}{l^2}\Big).
\end{aligned}$$

We define a general commutative model to be:

$$\begin{aligned} S &= c_1 S_1 + c_2 S_2 + c_3 S_3 \\ &= -\frac{1}{16\pi G_N}\int d^4x \Big(c_1 \frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd} R_{\mu\nu}^{\ \ ab} R_{\rho\sigma}^{\ \ cd} \\ &\quad + \sqrt{-g}\big((c_1+c_2)R - \frac{6}{l^2}(c_1 + 2c_2 + 2c_3)\big)\Big), \end{aligned} \qquad (17)$$

with $e_\mu{}^a = \frac{1}{l}\omega_\mu^{a5}$, $\sqrt{-g} = \det e_\mu^a$, $R = R_{\mu\nu}^{\ \ ab} e_a{}^\mu e_b{}^\nu$. The constants c_1, c_2 and c_3 are arbitrary and can be determined from $c_1 + c_2 = 1$, and the cosmological constant is given by

$$\Lambda = -3\frac{1 + c_2 + 2c_3}{l^2} \ .$$

Note that the cosmological constant Λ can be positive, negative or zero, regardless of the symmetry of our model. In this action the vielbeins and spin connection are independent variables. Varying the action with respect to the spin connection we obtain an equation which relates connection and vielbeins. From vielbeins we can construct the metric tensor

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \ . \tag{18}$$

The metric

$$g'_{\mu\nu} = \eta_{AB} D_\mu \Phi^A D_\mu \Phi^B \tag{19}$$

in the 'gauge' $\Phi^5 = l, \Phi^a = 0$ becomes $g_{\mu\nu}$.

The commutative action is invariant under the Lorentz gauge transformations by construction. In addition this action posses invariance under general coordinate transformations. This action will be our starting point for the construction of a noncommutative gravity theory.

References:

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Seiberg-Witten map

We work in the $\theta-$constant space or canonical NC space. The canonical NC can be introduced by replacing the usual product by the Moyal-Weyl $\star$ product

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x} \ , \qquad (20)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.
It is a small deformation parameter.

$$[x^\mu \overset{\star}{,} x^\nu] = i\theta^{\mu\nu}$$

The commutative quantities replace by their noncommutative counterparts.

$$
\begin{aligned}
\epsilon,\ \Phi,\ \Psi,\ \omega_\mu, &\quad\rightarrow\quad \hat{\Lambda}_\epsilon,\ \hat{\Phi},\ \hat{\Psi},\ \hat{\omega}_\mu, \\
F_{\mu\nu} &\quad\rightarrow\quad \hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \overset{\star}{,} \hat{\omega}_\nu] \\
\delta_\epsilon \Psi = i\epsilon\Psi &\quad\rightarrow\quad \delta^\star_\epsilon \hat{\Psi} = i\hat{\Lambda}_\epsilon \star \hat{\Psi} \\
\delta_\epsilon \Phi = i[\epsilon, \Phi] &\quad\rightarrow\quad \delta^\star_\epsilon \hat{\Phi} = i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{\Phi}] \\
\delta_\epsilon \omega_\mu = \partial_\mu \epsilon + i[\epsilon, \omega_\mu] &\quad\rightarrow\quad \delta^\star_\epsilon \hat{\omega}_\mu = \partial_\mu \hat{\Lambda}_\epsilon + i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{\omega}_\mu] \\
\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] &\quad\rightarrow\quad \delta^\star_\epsilon \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \overset{\star}{,} \hat{F}_{\mu\nu}]
\end{aligned}
$$

Commutative and noncommutative symmetries which correspond to the same gauge group can be related by the Seiberg-Witten map: the map enables one to express the noncommutative variables in terms of the commutative variables. In that way no new degrees of freedom are introduced. SW map can also be seen as an expansion in $\theta^{\mu\nu}$, so the SW approach is known as a $\theta-$expanded theory.

The noncommutative quantities $\hat{\Lambda}_\epsilon, \hat{\omega}_\mu, \hat{\Phi}$ are power series in the noncommutative parameter $\theta^{\mu\nu}$:

$$\begin{aligned} \hat{\Lambda}_\epsilon &= \epsilon + \hat{\Lambda}^{(1)} + \hat{\Lambda}^{(2)} + \dots \ , \\ \hat{\omega}_\mu &= \omega_\mu + \hat{\omega}_\mu^{(1)} + \hat{\omega}_\mu^{(2)} + \dots \ , \end{aligned} \qquad (21)$$

where the higher order corrections are functions of the commutative variables ϵ, ω_μ, and their derivatives. The requirement that the commutator of two deformed gauge transformations is a deformed transformation again:

$$[\delta^\star_\alpha \stackrel{\star}{,} \delta^\star_\beta] = \delta^\star_{-i[\alpha,\beta]} \qquad (22)$$

gives the solution for $\Lambda_\epsilon^{(1)}, \Lambda_\epsilon^{(2)}, \dots$.

Solving the equation

$$\hat{\omega}_\mu(\omega) + \delta^\star_\epsilon \hat{\omega}_\mu(\omega) = \hat{\omega}_\mu(\omega + \delta_\epsilon \omega) \tag{23}$$

order by order in the noncommutative parameter we can express noncommutative gauge potential $\hat{\omega}_\mu$ in terms of the commutative one. The first order solution is

$$\hat{\omega}^{(1)}_\mu = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda \omega_\mu + F_{\lambda\mu}\} \tag{24}$$

$$\widehat{\omega}_\mu = \omega_\mu - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta \omega_\mu + F_{\beta\mu}\} + \mathcal{O}(\theta^2) \, , \tag{25}$$

$$\widehat{\phi} = \phi - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)\phi\} + \mathcal{O}(\theta^2) \, , \tag{26}$$

$$\widehat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2) \, , \tag{27}$$

$$\widehat{\bar{\psi}} = \bar{\psi} - \frac{1}{4}\theta^{\alpha\beta}(\partial_\beta + D_\beta)\bar{\psi}\omega_\alpha + \mathcal{O}(\theta^2) \, , \tag{28}$$

$$\widehat{\Lambda}_\epsilon = \epsilon - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta \epsilon\} + \mathcal{O}(\theta^2) \, . \tag{29}$$

From potential $\hat{\omega}_\mu$ we find the field strength

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\omega}_\nu] \ , \tag{30}$$

The first order correction is

$$\hat{F}^{(1)}_{\mu\nu} = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu}\} + \frac{1}{2}\theta^{\kappa\lambda}\{F_{\mu\kappa}, F_{\nu\lambda}\} \tag{31}$$

Transformation low

$$\delta^\star_\epsilon \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{F}_{\mu\nu}] \tag{32}$$

NC Gravity Action

The NC generalization of (17) is given by

$$S_{NC} = c_1 S_{1NC} + c_2 S_{2NC} + c_3 S_{3NC} \,, \qquad (33)$$

with

$$\begin{aligned} S_{1NC} &= \frac{il}{64\pi G_N} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi} \,, \\ S_{2NC} &= \frac{1}{64\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \hat{\phi} \star \hat{F}_{\mu\nu} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} + c.c. \,, \\ S_{3NC} &= -\frac{i}{128\pi G_N l} \mathrm{Tr} \int \mathrm{d}^4 x \, \varepsilon^{\mu\nu\rho\sigma} D_\mu \hat{\phi} \star D_\nu \hat{\phi} \star \hat{D}_\rho \hat{\phi} \star \hat{D}_\sigma \hat{\phi} \star \hat{\phi} \,. \end{aligned}$$

It is invariant under the NC $SO(2,3)_\star$ gauge symmetry and the SW map guarantees that after the expansion it will be invariant under the commutative $SO(2,3)$ gauge symmetry.

Seiberg-Witten expansion

$$S_{NC} = S^{(0)} + S^{(1)} + S^{(2)} + \dots \ . \tag{34}$$

$$S^{(1)}_{NC} = 0$$

After the symmetry breaking the field $\phi^a = 0, \ \phi^5 = l$. We are interested in the low energy expansion we keep only the terms of the zeroth, the first and the second order in the derivatives of vierbeins (linear in $R_{\alpha\beta\gamma\delta}$, quadratic in $T^a_{\alpha\beta}$):

$$
\begin{aligned}
S_{NC} = \quad & \frac{1}{128\pi G_N l^4}\int \mathrm{d}^4x\, e\theta^{\alpha\beta}\theta^{\gamma\delta}\Big((-2+12c_2+38c_3)R_{\alpha\beta\gamma\delta} \\
& +(4-18c_2-44c_3)R_{\alpha\gamma\beta\delta}-(6+22c_2+36c_3)g_{\beta\delta}R_{\alpha\gamma}+\frac{6+28c_2+56c_3}{l^2}g_{\alpha\gamma}g_{\beta\delta} \\
& +(5-\frac{9}{2}c_2-7c_3)T^a_{\alpha\beta}T_{\gamma\delta a}+(-10+\frac{9}{2}c_2+14c_3)T^a_{\alpha\gamma}T_{\beta\delta a}+(3-3c_2-2c_3)T_{\alpha\beta\gamma}T_{\delta\mu}^{\ \ \mu} \\
& +(1+2c_2)T_{\alpha\beta\rho}T^{\rho}_{\ \gamma\delta}+8T_{\alpha\gamma\delta}T_{\beta\mu}^{\ \ \mu}-(2c_2+4c_3)T_{\alpha\gamma\rho}T^{\rho}_{\ \delta\beta} \\
& +(2c_2+4c_3)g_{\beta\delta}T_{\gamma\sigma}^{\ \ \sigma}T_{\alpha\rho}^{\ \ \rho}-(2c_2+4c_3)T_{\alpha\rho\sigma}T_{\gamma}^{\ \sigma\rho}g_{\beta\delta}+(-2+4c_2+18c_3)T_{\alpha\beta\gamma}e^{\rho}_a\nabla_\delta e^a_\rho \\
& +(6-8c_2-8c_3)T_{\alpha\gamma\beta}e^{\rho}_a\nabla_\delta e^a_\rho+(2+4c_2+12c_3)T_{\alpha\gamma}^{\ \ \mu}e^a_\beta\nabla_\delta e^a_\mu-T_{\alpha\beta}^{\ \ \mu}e^a_\delta\nabla_\gamma e^a_\mu \\
& +(-6-8c_2-16c_3)T_{\delta\rho\beta}e^{\rho}_a\nabla_\alpha e^a_\gamma-(2c_2+4c_3)g_{\alpha\gamma}T_{\mu\beta}^{\ \ \mu}e^{\rho}_a\nabla_\delta e^a_\rho-(2c_2+4c_3)g_{\beta\delta}T_{\alpha\rho}^{\ \ \sigma}e^{\rho}_a\nabla_\gamma e^a_\sigma \\
& -(4+16c_2+32c_3)e^{\mu}_a e_{b\beta}\nabla_\gamma e^a_\alpha\nabla_\delta e^b_\mu+(4+12c_2+32c_3)e_{\delta a}e^{\mu}_b\nabla_\alpha e^a_\gamma\nabla_\beta e^b_\mu \\
& -(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\nu}_b\nabla_\gamma e^a_\mu\nabla_\alpha e^b_\nu+(2+4c_2+8c_3)g_{\beta\delta}e^{\mu}_a e^{\rho}_c\nabla_\alpha e^a_\rho\nabla_\gamma e^c_\mu\Big).
\end{aligned}
\qquad (35)
$$

Properties:

-in the zeroth order the action (35) reduces to EH action and the cosmological constant term (arbitrary, constants c_2 and c_3).

-symmetries: NC generalization preserves local Lorentz symmetry but breaks the diffeomorphism symmetry. For example:
-$\nabla_\alpha e_\gamma^a$ can be written (using the metricity condition) as

$$\nabla_\alpha e_\gamma{}^a = \partial_\alpha e_\gamma{}^a + \omega_\alpha^{ab} e_{\gamma b} = \Gamma_{\alpha\gamma}^\sigma e_\sigma{}^a \tag{36}$$

-opservation: models with $SO(1,3)$ gauge symmetry cannot have a correction term of the form $\theta^{\alpha\beta}\theta^{\gamma\delta}g_{\alpha\gamma}g_{\beta\delta}$ (important for NC Minkowski corrections). Consequence of having the gauge field (not the vierbein) as the fundamental field.
-equations of motion: variation with respect to the vierbeins and the spin connection

$$\begin{aligned}
\delta e_\mu^a : \quad & R_{\alpha\gamma}{}^{cd} e_d^\gamma e_a^\alpha e_c^\mu - \frac{1}{2} e_a^\mu R + \frac{3}{l^2}(1 + c_2 + 2c_3) e_a^\mu \\
& = \tau_a{}^\mu = -\frac{8\pi G_N}{e} \frac{\delta S_{NC}^{(2)}}{\delta e_\mu^a}, \qquad (37) \\
\delta\omega_\mu^{ab} : \quad & T_{ac}{}^c e_b^\mu - T_{bc}{}^c e_a^\mu - T_{ab}{}^\mu \\
& = S_{ab}{}^\mu = -\frac{16\pi G_N}{e} \frac{\delta S_{NC}^{(2)}}{\delta \omega_\mu^{ab}}. \qquad (38)
\end{aligned}$$

NC corrections to Minkowski space-time

Minkowski space-time is a solution of vacuum Einstein equations without the cosmological constant $(1 + c_2 + 2c_3 = 0)$. We are interested in corrections to this solution induced by our NC gravity model.

We assume that the NC metric is of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is a small correction that is second order in the deformation parameter $\theta^{\mu\nu}$.

Inserting this ansatz into the action (35) and varying with respect to $h_{\mu\nu}$ leads to:

$$\begin{aligned} & \frac{1}{2}(\partial_\sigma\partial^\nu h^{\sigma\mu} + \partial_\sigma\partial^\mu h^{\sigma\nu} - \partial^\mu\partial^\nu h - \Box h^{\mu\nu}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h) \\ = \ & \frac{11}{4l^6}(2\eta_{\alpha\gamma}\theta^{\alpha\mu}\theta^{\gamma\nu} + \frac{1}{2}\eta_{\alpha\gamma}\eta_{\beta\delta}\eta^{\mu\nu}\theta^{\alpha\beta}\theta^{\gamma\delta})\,. \end{aligned} \tag{39}$$

The RHS of equation (39) is constant. Therefore, these equations are solved by a general $h_{\mu\nu}$ quadratic in coordinates. A solution of the form:

$$\begin{aligned} g_{00} &= 1 - \frac{11}{2l^6}\theta^{0m}\theta^{0n}x^m x^n - \frac{11}{8l^6}\theta^{\alpha\beta}\theta_{\alpha\beta}r^2 \\ g^{0i} &= -\frac{11}{3l^6}\theta^{0m}\theta^{0n}x^m x^n, \\ g_{ij} &= -\delta_{ij} - \frac{11}{6l^6}\theta^{im}\theta^{jn}x^m x^n \\ &\quad + \frac{11}{24l^6}\delta^{ij}\theta^{\alpha\beta}\theta_{\alpha\beta}r^2 - \frac{11}{24l^6}\theta^{\alpha\beta}\theta_{\alpha\beta}x^i x^j \,. \end{aligned} \tag{40}$$

Scalar curvature of this solution is
$R = -\frac{11}{l^6}\theta^{\alpha\beta}\theta^{\gamma\delta}\eta_{\alpha\gamma}\eta_{\beta\delta} = const.$, (A)dS-like solution. Curvature is induced by the noncommutativity.

The Reimann tensor for this solution can be calculated easily. A very interesting (and unexpected) observation follows: knowing the components of the Riemann tensor the components of the metric tensor can be written as

$$\begin{aligned} g_{00} &= 1 - R_{0m0n} x^m x^n \,, \\ g_{0i} &= -\frac{2}{3} R_{0min} x^m x^n \,, \\ g_{ij} &= -\delta_{ij} - \frac{1}{3} R_{imjn} x^m x^n \,. \end{aligned} \tag{41}$$

This shows that the coordinates x^μ we started with, are Fermi normal coordinates.

Riemann normal coordinates: inertial coordinates in a point, can be constructed in a small neighborhood of that point.
Fermi normal coordinates: inertial coordinates of a local observer moving along some geodesic; can be constructed in a small neighborhood along the geodesic (cylinder), [Manasse, Misner '63; Chicone, Mashoon'06; Klein, Randles '11].

The measurements performed by the local observer moving along the geodesic are described in the Fermi normal coordinates. Especially, he is the one that measures $\theta^{\mu\nu}$ to be constant! In any other reference frame, observers will measure $\theta^{\mu\nu}$ different from constant.

The breaking of diffeomorphism invariance is now understood better: there is a preferred reference system defined by the Fermi normal coordinates and the NC parameter $\theta^{\mu\nu}$ is constant in that particular reference system. The breaking Diff. symmetry can be a consequence of fixing the coordinate system.

Let y^α be an arbitrary coordinate system at a point P in a small neighborhood of the geodesic γ which defines our FNC x^μ and $[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$. Then the noncommutativity in y-coordinates is given by

$$[y^\alpha \stackrel{\star}{,} y^\beta] = i\theta^{\mu\nu}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} - \frac{i}{24}\theta^{\mu\nu}\theta^{\rho\sigma}\theta^{\kappa\lambda}\frac{\partial^3 y^\alpha}{\partial x^\kappa \partial x^\rho \partial x^\mu}\frac{\partial^3 y^\beta}{\partial x^\lambda \partial x^\sigma \partial x^\nu} + \dots . \tag{42}$$

The $\star$-product is the Moyal-Weyl $\star$-product and y^α are understood as functions of FNC x^μ.

Following closely the notation of [Poisson, Pound, Vega, arXiv:1102.0529][1] we calculate

$$
\begin{aligned}
\frac{\partial y^\beta}{\partial x^0} &= \bar{e}^\beta_0 - \frac{1}{2}\bar{e}^\beta_A R^A{}_{i0j} x^i x^j + \dots \\
\frac{\partial y^\beta}{\partial x^k} &= \bar{e}^\beta_k - \frac{1}{6}\bar{e}^\beta_A R^A{}_{ikj} x^i x^j + \dots . \qquad (43)
\end{aligned}
$$

Here $\bar{e}^\beta_A$ are vierbeins relating coordinates y^α and locally flat coordinates in the given point P and $R^A{}_{ikj}$ and $R^A{}_{i0j}$ are components of the Riemann tensor calculated at the geodesic γ (depending only on the affine parameter t along γ.). Equations (43) contain terms that are higher power in coordinates x^i (indices $i, j, k, \dots$ are spacial indices) and derivatives of Riemann tensor. We only wrote the first approximation.

[1] Note that we use the $(+, -, -, -)$ signature.

Using (43) we calculate the first term in (42):

$$
\begin{aligned}
i\theta^{\mu\nu}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} &= i\Big(\theta^{0i}(\bar{e}_0^\alpha\bar{e}_i^\beta - \bar{e}_i^\alpha\bar{e}_0^\beta) + \theta^{ij}\bar{e}_i^\alpha\bar{e}_j^\beta\Big) \\
&\quad -\frac{i}{6}\theta^{0k}(\bar{e}_0^\alpha\bar{e}_A^\beta - \bar{e}_A^\alpha\bar{e}_0^\beta)R^A{}_{ikj}x^i x^j \\
&\quad -\frac{i}{6}\theta^{kl}(\bar{e}_k^\alpha\bar{e}_A^\beta - \bar{e}_A^\alpha\bar{e}_k^\beta)R^A{}_{ilj}x^i x^j \\
&\quad +\frac{i}{2}\theta^{0k}(\bar{e}_k^\alpha\bar{e}_A^\beta - \bar{e}_A^\alpha\bar{e}_k^\beta)R^A{}_{i0j}x^i x^j + \ldots . \quad (44)
\end{aligned}
$$

Once the explicit form of y^α (in terms of FNC x^μ) is given, one can calculate $[y^\alpha \stackrel{\star}{,} y^\beta]$ more explicitly.

Dirac field coupled to gravity in NC $SO(2,3)_\star$ model

Kinetic term:

$$S_{kin} = \frac{i}{12}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\Big[\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi - D_\sigma\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\psi\Big]\,. \quad (45)$$

After breaking the symmetry we arrive at

$$S_{kin} = \frac{i}{2}\int d^4x\, e\, \left[\bar{\psi}\gamma^\sigma\nabla_\sigma\psi - \nabla_\sigma\bar{\psi}\gamma^\sigma\psi\right] - \frac{2}{l}\int d^4x\, e\, \bar{\psi}\psi\ , \quad (46)$$

which is exactly the Dirac action in curved space-time for spinors of mass $2/l$. But, we know that leptons and quarks do not have the same masses. Additional mass term

$$S_m = S_1 + S_2 + S_3$$

where

$$
\begin{aligned}
S_{1,m} &= \frac{i}{2}c_1\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi + c.c\ , \\
S_{2,m} &= \frac{i}{2}c_2\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi + c.c.\ , \\
S_{3,m} &= i\;c_3\Big(\frac{m}{l}-\frac{2}{l^2}\Big)\int d^4x\;\varepsilon^{\mu\nu\rho\sigma}\;\;\bar{\psi}D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi\ . \qquad (47)
\end{aligned}
$$

The coefficients c_1, c_2 and c_3 will be fixed later.

After the symmetry breaking, the sum of the mass terms (47) reduce to

$$
S_m = \sum_{i=1}^{3} S_{i,m} = 24(c_2 - c_1 - c_3)\left(m - \frac{2}{l}\right)\int d^4x\; e\;\bar{\psi}\psi\ . \qquad (48)
$$

In order to get the Dirac mass term the coefficients c_1, c_2, and c_3 must satisfy the following constraint:

$$c_2 - c_1 - c_3 = -\frac{1}{24} \ . \tag{49}$$

The noncommutative version of the kinetic action is

$$S_{NC} = \frac{i}{12} \int d^4x \ \varepsilon^{\mu\nu\rho\sigma} \ \Big[\widehat{\bar{\psi}} \star (D_\mu \widehat{\phi}) \star (D_\nu \widehat{\phi}) \star (D_\rho \widehat{\phi}) \star (D_\sigma \widehat{\psi}) \\ -(D_\sigma \widehat{\bar{\psi}}) \star (D_\mu \widehat{\phi}) \star (D_\nu \widehat{\phi}) \star (D_\rho \widehat{\phi}) \star \widehat{\psi}\Big] \ . \tag{50}$$

Example of calculation:

$$D_\mu\widehat{\phi} = D_\mu\phi - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)D_\mu\phi\} + \frac{1}{2}\theta^{\alpha\beta}\{F_{\alpha\mu}, D_\beta\phi\} + \mathcal{O}(\theta^2)\ , \tag{51}$$

$$D_\mu\widehat{\psi} = D_\mu\psi - \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)D_\mu\psi + \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\mu}D_\beta\psi + \mathcal{O}(\theta^2)\ . \tag{52}$$

$$\begin{aligned}(D_\mu\phi \star D_\nu\psi)^{(1)} &= (D_\mu\phi)^{(1)}D_\nu\psi + D_\mu\phi(D_\nu\psi)^{(1)} \\ &+ \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha(D_\mu\phi)\partial_\beta(D_\nu\psi) \\ &= -\frac{1}{4}\theta^{\alpha\beta}(\partial_\beta + D_\beta)(D_\mu\phi D_\nu\psi) \\ &+ \frac{i}{2}\theta^{\alpha\beta}(D_\alpha D_\mu\phi)(D_\beta D_\nu\psi) + \dots \end{aligned} \tag{53}$$

NC correction of kinetic action in the first order at θ is

$$
\begin{aligned}
S^{(1)}_{kin} = \frac{i}{12}\,\theta^{\alpha\beta}\int d^4x\,\varepsilon^{\mu\nu\rho\sigma}\Big[& -\frac{1}{4}\bar{\psi}F_{\alpha\beta}D_\mu\phi D_\nu D_\rho\phi D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi D_\rho\phi)(D_\beta D_\sigma\psi) \\
& + \frac{i}{2}\bar{\psi}D_\alpha(D_\mu\phi D_\nu\phi)(D_\beta D_\rho\phi)D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}(D_\alpha D_\mu\phi)(D_\beta D_\nu\phi)D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}\{F_{\alpha\mu}, D_\beta\phi\}D_\nu\phi D_\rho\phi D_\sigma\psi \\
& + \frac{1}{2}\bar{\psi}D_\mu\phi\{F_{\alpha\nu}, D_\beta\phi\}D_\rho\phi D_\sigma\psi \\
& + \frac{i}{2}\bar{\psi}D_\mu\phi D_\nu\phi\{F_{\alpha\rho}, D_\beta\phi\}D_\sigma\psi \\
& - \frac{1}{2}\bar{\psi}D_\mu\phi D_\nu\phi D_\rho\phi F_{\sigma\alpha}D_\beta\psi \Big] + h.c. \,. \qquad (54)
\end{aligned}
$$

This action possesses ordinary $SO(2,3)$, i.e. AdS symmetry. Taking $\phi^a = 0$ and $\phi^5 = l$, this symmetry is broken down to the local Lorentz symmetry. After the symmetry breaking, the first order kinetic term becomes

$$
\begin{aligned}
S^{(1)}_{kin} \quad = \quad & \theta^{\alpha\beta} \int d^4x \; e \Big[-\frac{1}{8} R_{\alpha\mu}{}^{ab} e^{\mu}_{a} \; \bar{\psi}\gamma_b \nabla_\beta \psi + \frac{1}{16} R_{\alpha\beta}{}^{ab} e^{\sigma}_{b} \; \bar{\psi}\gamma_a \nabla_\sigma \psi \\
- \quad & \frac{i}{32} R_{\alpha\beta}{}^{ab} \varepsilon_{abc}{}^{d} e^{\sigma}_{d} \; \bar{\psi}\gamma^c \gamma^5 \nabla_\sigma \psi - \frac{i}{16} R_{\alpha\mu}{}^{bc} e^{\mu}_{a} \varepsilon^{a}{}_{bcm} \; \bar{\psi}\gamma^m \gamma^5 \nabla_\beta \psi \\
- \quad & \frac{i}{24} \; R_{\alpha\mu}{}^{ab} \varepsilon_{abc}{}^{d} e^{c}_{\beta} (e^{\mu}_{d} e^{\sigma}_{s} - e^{\mu}_{s} e^{\sigma}_{d}) \; \bar{\psi}\gamma^s \gamma^5 \nabla_\sigma \psi \\
- \quad & \frac{i}{8l} T_{\alpha\beta}{}^{a} e^{\sigma}_{a} \; \bar{\psi} \nabla_\sigma \psi + \frac{i}{8l} T_{\alpha\mu}{}^{a} e^{\mu}_{a} \; \bar{\psi} \nabla_\beta \psi \\
+ \quad & \frac{1}{16l} T_{\alpha\beta}{}^{a} e^{\mu}_{a} \; \bar{\psi} \sigma_\mu{}^{\sigma} \nabla_\sigma \psi + \frac{1}{8l} T_{\alpha\mu}{}^{a} e^{\mu}_{b} \; \bar{\psi} \sigma_a{}^{b} \nabla_\beta \psi \\
- \quad & \frac{1}{4} (\nabla_\alpha e^{a}_{\mu})(e^{\mu}_{a} e^{\sigma}_{b} - e^{\sigma}_{a} e^{\mu}_{b}) \; \bar{\psi}\gamma^b \nabla_\beta \nabla_\sigma \psi - \frac{1}{4l} \bar{\psi} \sigma_\alpha{}^{\sigma} \nabla_\beta \nabla_\sigma \psi \\
- \quad & \frac{i}{8} \eta_{ab} (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu}) \varepsilon^{cdrs} e^{\mu}_{c} e^{\nu}_{d} e^{\sigma}_{s} \; \bar{\psi}\gamma_r \gamma_5 \nabla_\sigma \psi \\
+ \quad & \frac{i}{12} (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu}) \varepsilon_{b}{}^{cds} e^{\mu}_{c} e^{\nu}_{d} e^{\sigma}_{s} \; \bar{\psi}\gamma_a \gamma_5 \nabla_\sigma \psi \\
- \quad & \frac{1}{12l} e^{c}_{\alpha} (\nabla_\beta e^{b}_{\nu}) \varepsilon_{bc}{}^{ds} e^{\nu}_{d} e^{\sigma}_{s} \; \bar{\psi}\gamma_5 \nabla_\sigma \psi \\
- \quad & \frac{1}{8l} (\nabla_\alpha e^{a}_{\mu})(e^{\mu}_{a} e^{\sigma}_{b} - e^{\sigma}_{a} e^{\mu}_{b}) e^{c}_{\beta} \; \bar{\psi} \sigma^{b}{}_{c} \nabla_\sigma \psi \\
- \quad & \frac{i}{2l} (\nabla_\alpha e^{a}_{\mu}) e^{\mu}_{a} \; \bar{\psi} \nabla_\beta \psi - \frac{1}{8l} (\nabla_\alpha e^{a}_{\mu}) e^{\mu}_{b} \; \bar{\psi} \sigma_a{}^{b} \nabla_\beta \psi \\
+ \quad & \frac{1}{96l} R_{\alpha\beta}{}^{ab} \; \bar{\psi} \sigma_{ab} \psi - \frac{5}{48l} R_{\alpha\mu}{}^{ab} e^{\mu}_{a} e^{c}_{\beta} \; \bar{\psi} \sigma_{bc} \psi - \frac{1}{16l} R_{\alpha\mu}{}^{ab} e_{\beta a} e^{\mu}_{c} \; \bar{\psi} \sigma_b{}^{c} \psi \\
- \quad & \frac{3}{32l^2} T_{\alpha\beta}{}^{a} \; \bar{\psi}\gamma_a \psi - \frac{1}{16l^2} T_{\alpha\mu}{}^{a} e^{\mu}_{a} \; \bar{\psi}\gamma_\beta \psi \\
+ \quad & \frac{1}{16l^2} T_{\alpha\mu}{}^{a} e_{\beta a} \; \bar{\psi}\gamma^\mu \psi + \frac{1}{12l} \eta_{ab} (\nabla_\alpha e^{a}_{\mu})(\nabla_\beta e^{b}_{\nu}) \; \bar{\psi} \sigma^{\mu\nu} \psi
\end{aligned}
$$

NC correction to the mass term are calculated similarly.
The NC action is invariant under $SO(1,3)$ transformation and the charge conjugation.
There is a modification of Dirac eq. in flat space-time.

$$S_{NC} = \int d^4x \, \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \theta^{\alpha\beta} \int d^4x \left[-\frac{1}{2l}\bar{\psi}\sigma_\alpha{}^\sigma \partial_\beta \partial_\sigma \psi + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\bar{\psi}\gamma_\rho\gamma_5\partial_\sigma\psi - M\bar{\psi}\sigma_{\alpha\beta}\psi \right] . \tag{56}$$

The Feynman propagator

$$
\begin{aligned}
iS_F(p) &= \int d^4x \, \langle\Omega| T\psi(x)\bar{\psi}(0)|\Omega\rangle e^{ipx} \\
&= \frac{i}{\not{p} - m + i\epsilon} + \frac{i}{\not{p} - m + i\epsilon}(i\theta^{\alpha\beta} D_{\alpha\beta})\frac{i}{\not{p} - m + i\epsilon} + \dots \ ,
\end{aligned}
$$

where

$$
D_{\alpha\beta} := \frac{1}{2l}\sigma_{\alpha}{}^{\sigma} p_\beta p_\sigma + \frac{7}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5 p_\sigma - M\sigma_{\alpha\beta} \ . \tag{57}
$$

The correction of dispersion relation is the second order in θ.

Electromagnetic field

Gauge group: $SO(2,3) \otimes U(1)$.
Gauge potential

$$\Omega_\mu = \omega_\mu + A_\mu \ . \tag{58}$$

The field strength associated with the gauge potential Ω_μ is

$$\mathbb{F}_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu - i[\Omega_\mu, \Omega_\nu] \ , \tag{59}$$

and it can be decomposed as

$$\mathbb{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{F}_{\mu\nu} \ , \tag{60}$$

Additional term in the action

$$S_C = -\frac{1}{8l} Tr \int d^4x \ \varepsilon^{\mu\nu\rho\sigma} \Big[f \mathbb{F}_{\mu\nu} D_\rho \phi D_\sigma \phi \phi + \frac{i}{3!} f f D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi \Big] \ .$$

Work in progress.

Conclusion

NC gravity: a general NC $SO(2,3)_\star$ action studied, expansion up to second order in the NC parameter written in a manifestly gauge covariant way;

NC corrections to Minkowski space-time

emergence of Fermi normal coordinates

better understanding of θ-constant noncomutativity (deformation of black hole solution,..)

Coupling Dirac and electromagnetic field with gravity