T-duality and non-geometry

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9th MATHEMATICAL PHYSICS MEETING 2017
Belgrade, Serbia

Outline

- Closed string T-duality
- Open string T-duality
  - What is T-dual to local gauge transformations?
  - T-dual background fields of the open string
  - Relation with standard approach
- Non-geometric theories

Closed string

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}[x] + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}[x]\Big] \partial_\alpha x^\mu \partial_\beta x^\nu$$

Action principle $\delta S = 0$ gives equations of motion and boundary conditions

$$\gamma_\mu^{(0)}(x)\delta x^\mu /_{\sigma=\pi} \quad - \quad \gamma_\mu^{(0)}(x)\delta x^\mu /_{\sigma=0} = 0$$

where we define σ-momentum

$$\gamma_\mu^{(0)}(x) \equiv \frac{\delta S}{\delta x'^\mu} = \kappa\Big(2B_{\mu\nu}\dot{x}^\nu - G_{\mu\nu}x'^\nu\Big)$$

Buscher T-duality procedure

- Buscher procedure:
  - gauging global symmetries $\delta x^\mu = \lambda^\mu$
    $\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha$,
    - v^μ_α gauge field
    - D_α covariant derivative
  - Field strength $F^\mu_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha$
  - T-dual theory must be Physically equivalent to initial theory
    $F^\mu_{01} \equiv F^\mu = 0$

Buscher T-duality procedure-1

- Invariant Action

$$S_{inv}(x,y,v) = \frac{1}{2\pi\alpha'}\int\Big[(\frac{\eta^{\alpha\beta}}{2}G_{\mu\nu} + \varepsilon^{\alpha\beta}B_{\mu\nu})D_\alpha x^\mu D_\beta x^\nu \; + \frac{\alpha'}{2}y_\mu F^\mu\Big]$$

  - y_μ Lagrange multiplier
- Gauge fixing $x^\mu = 0$
- Gauge fixed Action

$$S_{fix}(y,v) = \frac{1}{2\pi\alpha'}\int\Big[(\frac{\eta^{\alpha\beta}}{2}G_{\mu\nu} + \varepsilon^{\alpha\beta}B_{\mu\nu})v_\alpha^\mu v_\beta^\nu \; + \frac{\alpha'}{2}y_\mu F^\mu\Big]$$

Buscher T-duality procedure-2

- Check
  y_μ: $\partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha = 0 \Longrightarrow v^\mu_\alpha = \partial_\alpha x^\mu \Longrightarrow S_{fix} \to S(x)$
- Elimination of gauge fields on equations of motion produces T-dual Action

$$^\star S(y) = \frac{1}{2\pi\alpha'} \int_\Sigma \left(\frac{\eta^{\alpha\beta}}{2} {}^\star G^{\mu\nu} + \varepsilon^{\alpha\beta\star} B^{\mu\nu}\right) \partial_\alpha y_\mu \partial_\beta y_\nu \,,$$

Buscher T-duality procedure-3

- Dual Action ${}^\star S(y)$ has the same form as initial one, but with different background fields

$$ {}^\star S[y] = \kappa \int d^2\xi\ \partial_+ y_\mu\, {}^\star\Pi_+^{\mu\nu}\, \partial_- y_\nu = \frac{\kappa^2}{2} \int d^2\xi\ \partial_+ y_\mu \theta_-^{\mu\nu} \partial_- y_\nu $$

$$ {}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu} $$

where T-dual background fields

$$ G^E_{\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \qquad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1} B G^{-1})^{\mu\nu} $$

$$ \Pi_\pm \equiv B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}\,, \qquad \theta_\pm^{\mu\nu} \equiv \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu} $$

T-duality transformation of variables

- T-dual transformations

$$v_\pm^\mu \cong \partial_\pm x^\mu \cong -\kappa \Theta_\pm^{\mu\nu} \partial_\pm y_\nu$$

- together with inverse transformation produces
  T-duality transformation of variables

$$\partial_\pm x^\mu \cong -\kappa \theta_\pm^{\mu\nu} \partial_\pm y_\nu \,, \qquad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu} \partial_\pm x^\nu$$

- in canonical form

$$\kappa\, x'^\mu \cong {}^\star\pi^\mu \,, \quad \pi_\mu \cong \kappa y'_\mu \quad -\kappa\, \dot{x}^\mu \cong {}^\star\gamma^\mu_{(0)}(y) \,, \quad \gamma_\mu^{(0)}(x) \cong -\kappa\, \dot{y}_\mu$$

Open string T-duality

- Each term must have its own T-dual

$$
\begin{array}{ccccc}
S(x) & G_{\mu\nu} & B_{\mu\nu} & A^N_a & A^D_i \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
{}^\star S(y) & {}^\star G^{\mu\nu} & {}^\star B^{\mu\nu} & {}^\star A^a_D & {}^\star A^i_N
\end{array}
$$

- Coupling for Neumann fields

$$S_{A^N} = 2\kappa \int d\tau (A^N_a \dot{x}^a /_{\sigma=\pi} - A^N_a \dot{x}^a /_{\sigma=0})$$

- Coupling for Dirichlet fields

$$S_{A^D} = 2\kappa \int d\tau (A^D_i (?)^i /_{\sigma=\pi} - A^D_i (?)^i /_{\sigma=0})$$

Zwiebach approach

- Action of closed string theory is invariant under local gauge transformations

$$\delta_\Lambda G_{\mu\nu} = 0\,, \qquad \delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

- The open string theory is not invariant

$$\delta_\Lambda S[x] = 2\kappa \int d\tau (\Lambda_a \dot{x}^a /_{\sigma=\pi} - \Lambda_a \dot{x}^a /_{\sigma=0})$$

- To obtain gauge invariant action we should add the term

$$S_{A^N}[x] = 2\kappa \int d\tau (A^N_a \dot{x}^a /_{\sigma=\pi} - A^N_a \dot{x}^a /_{\sigma=0})$$

where newly introduced vector field A^N_a transforms with the same gauge parameter Λ_a

$$\delta_\Lambda A^N_a = -\Lambda_a$$

What is T-dual to local gauge transformations?

- If variation of energy-momentum tensor $T_\pm$ can be written as

$$\delta T_\pm = \{\Gamma, T_\pm\}$$

  then corresponding transformation of background fields is target-space symmetry of the theory.
- For $\Gamma \to \Gamma_\Lambda = 2\int d\sigma \Lambda_\mu \kappa x'^\mu$ we can obtain just local gauge transformations
- T-dual to $\kappa x'^\mu$ is π_μ so, T-dual to Γ_Λ is

$$\Gamma_\xi = 2\int d\sigma\, \xi^\mu \pi_\mu$$

  and corresponding transformations are

$$\delta_\xi G_{\mu\nu} = -2\left(D_\mu \xi_\nu + D_\nu \xi_\mu\right)$$

$$\delta_\xi B_{\mu\nu} = -2\,\xi^\rho B_{\rho\mu\nu} + 2\partial_\mu(B_{\nu\rho}\xi^\rho) - 2\partial_\nu(B_{\mu\rho}\xi^\rho)$$

What is T-dual to local gauge transformations?

- These transformations exactly have the form of general coordinate transformations (GCT) (symmetry transformations of the space-time action)
- Are they symmetries of the σ-model action?
- Action of σ-model is scalar under GCT, so both closed and open string actions are invariant under GCT

What is T-dual to local gauge transformations?

- it is useful to make
  - transformations of the background fields (metric tensor $G_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$) with parameter ξ_μ
  - the transformations of the string coordinates x^μ with different parameter $\delta x^\mu = \bar{\xi}^\mu$
  - Using the equation of motion we obtain

$$\delta_\xi S[x] = -2 \int_{\partial\Sigma} d\tau (\xi_\mu - \bar{\xi}_\mu)\, G^{-1\mu\nu} \gamma_\nu^{(0)}(x)$$

What is T-dual to local gauge transformations?

- Residual general coordinate transformations (RGCT) , which include the transformations of background fields but not include the transformations of the string coordinates x^μ
- $\bar{\xi}_\mu/_{\sigma=\pi} = \bar{\xi}_\mu/_{\sigma=0} = 0$

$$\delta_\xi S[x] = -2 \int_{\partial\Sigma} d\tau \xi_\mu \, G^{-1\mu\nu} \gamma_\nu^{(0)}(x)$$

- $\dot{x}^\mu$ and $\gamma_\mu^{(0)}(x)$ are expressions T-dual to each other
- local gauge transformations and RGCT are connected by T-duality
- strong indication that we are on the right track

Full gauge invariant action for open string

- Gauge invariant action for open string

$$S_{open}[x] = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu$$
$$+ \quad 2\kappa \int_{\partial\Sigma} d\tau \Big[A^N_a[x] \dot{x}^a - \frac{1}{\kappa} A^D_i[x] G^{-1ij} \gamma^{(0)}_j(x) \Big]$$

where

$$\delta_\xi A^D_i = -\xi_i$$

- In literature
  - $A^N_a[x]$ is known as massless vector field on Dp-brane
  - $A^D_i[x]$ is known as massless scalar oscillations orthogonal to the Dp-brane

Gauge invariant-physical variables

- Gauge invariant and physical variables

$$\begin{aligned}\mathcal{B}_{ab} &= B_{ab} + F^{(a)}_{ab}\,, \qquad \mathcal{G}_{ab} = G_{ab} \\ \mathcal{B}_{ij} &= B_{ij} - 2A^k_D B_{kij} - F^{(a)}_{ij}(\hat{A}^D) \\ \mathcal{G}_{ij} &= G_{ij} + F^{(s)}_{ij}(A^D)\end{aligned}$$

- Field strengths

$$F^{(a)}_{ab} = \partial_a A^N_b - \partial_b A^N_a\,, \qquad F^{(s)}_{ij}(A^D) = -2(\partial_i A^D_j + \partial_j A^D_i)$$

$$\hat{A}_i = 2B_{ij}G^{-1jk}A^D_k$$

T-dual background fields of the open string

- Choose background Vector fields linear in coordinates

$$B_{\mu\nu} = const\,, \qquad G_{\mu\nu} = const$$

$$A^N_a(x) = A^0_a - \frac{1}{2}F^{(a)}_{ab}x^b\,, \qquad A^D_i(x) = A^0_i - \frac{1}{4}F^{(s)}_{ij}x^j$$

so that corresponding field strengths are constant

- These forms of background fields satisfies space-time equations of motion for open string

T-dual background fields of the open string

- Action depends on the coordinate x^μ itself and not only on its derivatives with respect to τ and σ,
- Part with $A_i^D(x)$ does not have global shift symmetry, because the expression $\gamma_i^{(0)}$ contain x'^j which is not total derivative with respect to integration variable τ.
- So, we should apply T-dualization procedure which work in absence of global symmetry
  Lj. Davidović and B. Sazdović, *JHEP* **11** (2015) 119

T-dual background fields of the open string

- T-dual background fields in terms of initial ones

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}$$

$$^\star A_D^a(V) = G_E^{-1ab} A_b^N(V)\,, \quad ^\star A_N^i(V) = G^{-1ij} A_j^D(V)$$

- T-duality interchange Neumann with Dirichlet gauge fields

$$V^\mu = -\kappa\,\theta^{\mu\nu} y_\nu + G_E^{-1\mu\nu}\,\tilde{y}_\nu$$

$$\tilde{y}_\mu \equiv -\varepsilon_\alpha{}^\beta \int d\xi^\alpha \partial_\beta y_\mu = \int (d\tau y'_\mu + d\sigma \dot{y}_\mu)$$

$$\dot{\tilde{y}}_\mu = y'_\mu\,, \qquad \tilde{y}'_\mu = \dot{y}_\mu$$

Relation with standard approach

- Up to gauge transformation

$$ {}^{\star}A_D^a = G_E^{-1ab}\left(A_b^N + \frac{1}{2}y_a\right)\,, \quad {}^{\star}A_N^i = G^{-1ij}A_j^D $$

- In standard approach one can not recognize Dirichlet vector fields. So $A_i^D = 0$ and ${}^{\star}A_D^a = 0$ and

$$ {}^{\star}A_N^i = 0\,, \qquad y_a = -2A_b^N $$

This is consistency of standard approach

The field strength for non-geometric theories

- The particular form of $V^\mu = -\kappa\, \theta^{\mu\nu} y_\nu + G_E^{-1\mu\nu}\, \tilde{y}_\nu$ implies features of non-geometric theories

  Lj. Davidović, B. Nikolić and B. Sazdović, *EPJ* **C 74** (2014) 2734

  Lj. Davidović, B. Nikolić and B. Sazdović, *EPJ* **C 75** (2015) 576

  It produces non-commutativity and non-associativity of closed string coordinates

- In geometric theories the field strength for Abelian vector field is simple $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

  Because in non-geometric theories the vector field depends on V^μ, we expect that T-dual field strength will contain derivatives with respect to both variables y_μ and $\tilde{y}_\mu$

The field strength for non-geometric theories

- How to define the field strength for non-geometric theories?

  For Neumann vector fields (initial theory)

$$S_A^N[x] = 2\kappa \int_{\partial\Sigma} d\tau A_a^N(x)\dot{x}^a = \kappa \int_{\Sigma} d^2\xi \partial_+ x^a \mathcal{F}_{ab}\, \partial_- x^b$$

  where only antisymmetric part contributes

$$\mathcal{F}_{ab} = F_{ab}^{(a)} = \partial_a A_b^N(x) - \partial_b A_a^N(x)$$

  We are going to generalize such relation to non-geometric theories

The field strength for non-geometric theories

For Dirichlet vector fields (initial theory)

$$S_A^D[x] = 2\kappa \int_{\partial\Sigma} d\tau \Big(-\frac{1}{\kappa} A_i^D(x) G^{-1ij} \gamma_j^{(0)}(x) \Big)$$

$$= 2\kappa \int_{\partial\Sigma} d\tau \Big(\mathcal{A}_{0i}[x]\dot{x}^i - \mathcal{A}_{1i}[x] x'^i \Big) = \kappa \int_\Sigma d^2\xi \, \partial_+ x^i \, \mathcal{F}_{ij} \, \partial_- x^j$$

Now, both antisymmetric and symmetric parts contribute

$$\mathcal{F}_{ij} = \mathcal{F}_{ij}^{(a)} + \frac{1}{2}\mathcal{F}_{ij}^{(s)}$$

where

$$\mathcal{F}_{ij}^{(a)} = \Big[\partial_i \Big(2B_{jk} G^{-1kq} A_q^D \Big) - \partial_j \Big(2B_{ik} G^{-1kq} A_q^D \Big) \Big]$$
$$= \partial_i \mathcal{A}_{0j}(x) - \partial_j \mathcal{A}_{0i}(x)$$

$$\mathcal{F}_{ij}^{(s)} = -2(\partial_i A_j^D + \partial_j A_i^D) = 2\Big(\partial_i \mathcal{A}_{1j}(x) + \partial_j \mathcal{A}_{1i}(x) \Big)$$

The field strength for non-geometric theories

- For Dirichlet vector fields (T-dual theory)

$$^\star S_A^D[y] = 2\kappa \int_{\partial\Sigma} d\tau \Big(-\frac{1}{\kappa} {}^\star A_D^a(V)\, {}^\star G^{-1}_{ab}\, {}^\star \gamma^b_{(0)}(y) \Big)$$

$$= \kappa \int_\Sigma d^2\xi \partial_+ y_a\, {}^\star \mathcal{F}^{ab}\, \partial_- y_b$$

$$^\star \mathcal{F}^{ab} = {}^\star \mathcal{F}^{ab}_{(a)} + \frac{1}{2} {}^\star \mathcal{F}^{ab}_{(s)}$$

- For Neumann vector fields (T-dual theory)

$$^\star S_A^N[y] = 2\kappa \int_{\partial\Sigma} d\tau \Big({}^\star A^i_N(V) \dot{y}_i \Big) = \kappa \int_\Sigma d^2\xi\, \partial_+ y_i\, {}^\star \mathcal{F}^{ij}\, \partial_- y_j$$

$$^\star \mathcal{F}^{ij} = {}^\star \mathcal{F}^{ij}_{(a)} + \frac{1}{2} {}^\star \mathcal{F}^{ij}_{(s)}$$

The field strength for non-geometric theories

Dirichlet

$$^\star\mathcal{F}^{ab}_{(a)} = -\kappa^2 \theta^{ac} F^{(a)}_{cd}\, \theta^{db} - G_E^{-1ac} F^{(a)}_{cd}\, G_E^{-1db}$$

$$^\star\mathcal{F}^{ab}_{(s)} = -2\kappa \Big[G_E^{-1ac} F^{(a)}_{cd}\, \theta^{db} + \theta^{ac} F^{(a)}_{cd}\, G_E^{-1db} \Big]$$

Neumann

$$^\star\mathcal{F}^{ij}_{(a)} = -\frac{\kappa}{4} \left(\theta^{ik} F^{(s)}_{kq} G^{-1qj} + G^{-1ik} F^{(s)}_{kq} \theta^{qj} \right)$$

$$^\star\mathcal{F}^{ij}_{(s)} = -\frac{1}{2} \left(G_E^{-1ik} F^{(s)}_{kq} G^{-1qj} + G^{-1ik} F^{(s)}_{kq} G_E^{-1qj} \right)$$

The field strength for non-geometric theories

- Write out expressions for T-dual field strengths ${}^\star\mathcal{F}^{\mu\nu}$ in terms of derivative of T-dual gauge fields ${}^\star\mathcal{A}_0^a(V)$ and ${}^\star\mathcal{A}_1^a(V)$ with respect to variables y_μ and $\tilde{y}_\mu$

$$ {}^\star\mathcal{F}_{(a)}^{\mu\nu} = \partial_y^\mu\, {}^\star\mathcal{A}_0^\nu(V) - \partial_y^\nu\, {}^\star\mathcal{A}_0^\mu(V) + \partial_{\tilde{y}}^\mu\, {}^\star\mathcal{A}_1^\nu(V) - \partial_{\tilde{y}}^\nu\, {}^\star\mathcal{A}_1^\mu(V)\,, $$

$$ {}^\star\mathcal{F}_{(s)}^{\mu\nu} = 2\left[\partial_{\tilde{y}}^\mu\, {}^\star\mathcal{A}_0^\nu(V) + \partial_{\tilde{y}}^\nu\, {}^\star\mathcal{A}_0^\mu(V) + \partial_y^{\mu\star}\mathcal{A}_1^\nu(V) + \partial_y^\nu\, {}^\star\mathcal{A}_1^\mu(V)\right] $$

- We can check this expression in other way

$$ {}^\star S_A[y] = {}^\star S_A^D[y] + {}^\star S_A^N[y] = 2\kappa\eta^{\alpha\beta}\int_{\partial\Sigma} d\tau^\star\mathcal{A}_\alpha^\mu[V]\,\partial_\beta y_\mu $$

$$ = \kappa\int_\Sigma d^2\xi\partial_+ y_\mu{}^\star\mathcal{F}^{\mu\nu}\partial_- y_\nu $$

The field strength for non-geometric theories

- The red expression we can consider as a general definition of the field strength
- Beside antisymmetric part ${}^\star\mathcal{F}^{\mu\nu}_{(a)}$ it also contains the symmetric one ${}^\star\mathcal{F}^{\mu\nu}_{(s)}$
- In definition of both parts, derivatives with respect to both T-dual coordinate y_μ and to its double $\tilde{y}_\mu$ contribute
- The unusual form of ${}^\star\mathcal{F}^{\mu\nu}$ is a consequence of two facts:
  1. the T-dual vector field ${}^\star A^a_D(V)$ are not multiplied by $\dot{y}_a$ but with T-dual σ-momentum ${}^\star G^{-1}_{ab}{}^\star\gamma^b_{(0)}$
  2. the T-dual vector fields depend on V^μ which is function on both y_μ and $\tilde{y}_\mu$