Hawking radiation by Schwarzschild-de Sitter black holes: fermionic fields

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This presentation is mainly based on our published paper:

- **Introduction**: Dirac eq. in curved spacetimes, Cartesian gauge
- Schwarzschild-de Sitter black holes
- Solutions to Dirac eq. in SdS geometry
- Analytical low energy SdS greybody factors
- Hawking radiation. Energy emission rate
- Conclusions
The Dirac equation

Introduction

The Dirac equation

\[ i \gamma^a D_a \psi - m \psi = 0 \]

it results from the gauge invariant action

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{i}{2} \bar{\psi} \gamma^a D_a \psi - \frac{i}{2} (\overline{D_a \psi}) \gamma^a \psi - m \bar{\psi} \psi \right\} \]

The correct covariant derivative

\[ D_a = \partial_a + \frac{i}{2} S^b_c \omega^c_{ab} \]

where \( \partial_a = e^\mu_a \partial_\mu \), with \( e^\mu_a \) the tetrad fields and \( S^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] \)

The spin-connection

\[ \omega^c_{ab} = e^\mu_a e^\nu_b \left( \hat{e}^c_\lambda \Gamma^\lambda_{\mu\nu} - \hat{e}^c_{\nu,\mu} \right) \]
The Dirac equation
Definition of the Cartesian gauge

- The explicit form

\[(i\gamma^a e^\mu_a \partial_\mu - m)\psi + \frac{i}{2} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e^\mu_a)\gamma^a \psi - \frac{1}{4} \{\gamma^a, S^b_c\} \omega^c_{ab} \psi = 0\]

- The tetrad fields \( \hat{e}^a(x) = \hat{e}^a_\mu dx^\mu \) (i.e. the 1-forms) defining the Cartesian gauge are

\[
\begin{align*}
\hat{e}^0 &= h(r)dt \\
\hat{e}^1 &= \frac{1}{h(r)} \sin \theta \cos \phi \, dr + r \cos \theta \cos \phi \, d\theta - r \sin \theta \sin \phi \, d\phi \\
\hat{e}^2 &= \frac{1}{h(r)} \sin \theta \sin \phi \, dr + r \cos \theta \sin \phi \, d\theta + r \sin \theta \cos \phi \, d\phi \\
\hat{e}^3 &= \frac{1}{h(r)} \cos \theta \, dr - r \sin \theta \, d\theta
\end{align*}
\]
The Dirac equation
Separation of variables

Particle-like energy eigenspinors of positive frequency and energy $E$ (I. I. Cotaescu, Mod. Phys. Lett. A 22, 2493, 2007)

$$\psi(x) = \psi_{E,j,m,\kappa}(t, r, \theta, \phi) = e^{-iEt} \frac{1}{r h(r)^{1/2}} \left[ F^+_{E,\kappa}(r) \Phi^+_{m_j,\kappa}(\theta, \phi) + F^-_{E,\kappa}(r) \Phi^-_{m_j,\kappa}(\theta, \phi) \right]$$

$F^\pm_{E,\kappa}(r)$ - radial wave functions.

$\Phi^\pm_{m_j,\kappa}(\theta, \phi)$ - usual four-component angular spinors.

The antiparticle-like energy eigenspinors can be obtained directly using the charge conjugation as in the flat case:

$$V_{E,j,m,\kappa} = (\psi_{E,j,m,\kappa})^c \equiv C(\bar{\psi}_{E,j,m,\kappa})^T, \quad C = i\gamma^2\gamma^0$$
SdS black holes

The Dirac equation

- The Schwarzschild-de Sitter line element

\[ ds^2 = h(r) \, dt^2 - \frac{dr^2}{h(r)} - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

\[ h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \]

- The radial Dirac equation for the upper component \( F^+(r) \) reads:

\[ \frac{d^2 F^+}{dx^2} + \left[ \epsilon^2 \left( \frac{1 - \lambda \sqrt{h}}{1 + \lambda \sqrt{h}} \right) + \frac{d}{dx} \left( \frac{k \sqrt{h}}{(1 + \lambda \sqrt{h})r} \right) - \frac{k^2 h}{(1 + \lambda \sqrt{h})^2 r^2} \right] F^+ = 0 \]

where:

\[ \frac{dr}{dx} = \frac{h}{1 + \lambda \sqrt{h}}, \quad \lambda = m/\epsilon \]
For \( r \to r_b \) and \( r \to r_c \) the function \( h(r) \to 0 \) and we obtain a more simple equation

\[
\frac{d^2 F^+}{dx^2} + \epsilon^2 F^+ = 0
\]

having general solutions of the form

\[
F^+(x) = A e^{-i\epsilon x} + B e^{i\epsilon x}
\]

Near the two horizons the new variable \( x \) behaves as:

\[
x \approx \begin{cases} 
\left( \frac{2M}{r_b^2} - \frac{2\Lambda}{3} r_b \right)^{-1} \ln h \equiv p \ln h, & \text{if } r \to r_b \\
\left( \frac{2M}{r_c^2} - \frac{2\Lambda}{3} r_c \right)^{-1} \ln h \equiv q \ln h, & \text{if } r \to r_c 
\end{cases}
\]
After imposing the ingoing boundary condition at the black hole horizon the solution in this (transition) region becomes:

\[ F_b^+ = A^{tr} e^{-i\epsilon x} \approx A^{tr} e^{-i\epsilon p \ln h} \]

At the cosmological horizon we have no restrictions, thus the solution will be a combination of ingoing and outgoing modes:

\[ F_c^+ = A^{in} e^{-i\epsilon x} + A^{out} e^{i\epsilon x} \approx A^{in} e^{-i\epsilon q \ln h} + A^{out} e^{i\epsilon q \ln h} \]
In this intermediate region $r_b < r < r_c$ the radial equation reduces to:

\[
\begin{align*}
\frac{d^2 F^+}{dx^2} + \frac{d}{dx} \left( \frac{k \sqrt{h}}{(1 + \lambda \sqrt{h}) r} \right) - \frac{k^2 h}{(1 + \lambda \sqrt{h})^2 r^2} \right) F^+ = 0
\end{align*}
\]

for which (after some calculations) we find the following solution:

\[
F^+_I = (A_2 + B_2 C(r)) F^+_{\text{hom}}
\]

where

\[
F^+_{\text{hom}} = H_0^{-1} (1 - \frac{\Lambda}{3} I)
\]

\[
H_0 = C \left( \frac{1 - \sqrt{h_0}}{1 + \sqrt{h_0}} \right)^{-k}, \quad h_0 = 1 - \frac{2M}{r}
\]
and also

\[ C(r) = \int \left( \frac{1 - \sqrt{h_0}}{1 + \sqrt{h_0}} \right)^{-2k} \left( \frac{1 + \frac{\Lambda}{3} I}{1 - \frac{\Lambda}{3} I} \right) \left( \frac{1}{h} + \frac{\lambda}{\sqrt{h}} \right) \frac{r^2}{2M - 2\Lambda/3} \frac{dh}{r^3} \]

\[ I = \frac{1}{2} \int \frac{kr}{(\sqrt{h_0})^3} \, dr = \frac{1}{2} \frac{k}{r - 2M} \left[ r\sqrt{h_0}(r^2 + 5Mr - 30M^2) \right. \]

\[ + \left. 15M^2(r - 2M) \ln(2r\sqrt{h_0} + 2r - 2M) \right] \]
SdS black holes
Solutions of Dirac equation in SdS

- Near the black hole
  \[ F_b^+ = A_{tr} e^{-i\epsilon x} \approx A_{tr} e^{-i\epsilon \log \hbar} \]

- in the intermediate region
  \[ F_l^+ = (A_2 + B_2 C(r)) F_{hom}^+ \]

- near the cosmological horizon
  \[ F_c^+ = A_{in} e^{-i\epsilon x} + A_{out} e^{i\epsilon x} \]
  \[ \approx A_{in} e^{-i\epsilon q \log \hbar} + A_{out} e^{i\epsilon q \log \hbar} \]
At low energies our solutions behave as

\[ F^+_b \approx A^{ir} (1 - i\epsilon p \ln h + ...) \]

\[ F^+_c \approx A^{in} (1 - i\epsilon q \ln h + ...) + A^{out} (1 + i\epsilon q \ln h + ...) \]

\[ \lim_{r \to r_b,c} F^+_I = \alpha_{b,c} (A_2 + \beta_{b,c} B_2 \ln h) \]

\[ \alpha_{b,c} = H_0^{-1} \left( 1 - \frac{\Lambda}{3} I \right) \bigg|_{r=r_b,c} \]

\[ \beta_{b,c} = \left( \frac{1 - \sqrt{h_0}}{1 + \sqrt{h_0}} \right)^{-2k} \left( \frac{1 + \frac{\Lambda}{3} I}{1 - \frac{\Lambda}{3} I} \right) \frac{r^2}{2M - 2\Lambda/3} \bigg|_{r=r_b,c} \]
Matching of the solutions

\[ F^+_b \approx A^{tr}(1 - i\epsilon p \ln h + \ldots) \]

\[ F^+_I \approx \alpha_b (A_2 + \beta_b B_2 \ln h) \]

The result (1)

\[ A_2 = \frac{1}{\alpha_b} A^{tr} \quad B_2 = -\frac{i\epsilon p}{\alpha_b \beta_b} A^{tr} \]
Matching of the solutions

\[ F_+^i \approx \alpha_b (A_2 + \beta_b B_2 \ln h) \]

\[ F_+^c \approx A^{in}_1 (1 - i\epsilon q \ln h + \ldots) + A^{out}_1 (1 + i\epsilon q \ln h + \ldots) \]

The result (2)

\[ A^{in} = \frac{\alpha_c}{2} \left( A_2 - \frac{\beta_c}{i\epsilon q} B_2 \right) \quad A^{out} = \frac{\alpha_c}{2} \left( A_2 + \frac{\beta_c}{i\epsilon q} B_2 \right) \]
Greybody Factors

The final result for the greybody factors (C.A. Sporea, A. Borowiec, IJMPD 25 (2016) 1650043)

\[ \Gamma_j(\epsilon) \equiv 1 - \left| \frac{A_{\text{out}}}{A_{\text{in}}} \right|^2 = 1 - \left( \frac{p \beta_c - q \beta_b}{p \beta_c + q \beta_b} \right)^2 \]

Comparing numerically the greybody factors (or equivalently the absorption cross section) in the massless limit for the lowest angular quantum numbers \( j = \frac{1}{2} \) for fermions, respectively \( s = 0 \) for scalars) we obtain that their ratio is approximatively.

\[ \frac{\Gamma_{j=\frac{1}{2}}}{\Gamma_{s=0}} \propto \frac{\sigma_{\text{abs}}^{j=\frac{1}{2}}}{\sigma_{\text{abs}}^{s=0}} \approx \frac{1}{12} \]

In the case of a Schwarzschild black hole the same ratio is equal with 1/8
Greybody Factors
Numerical examples

<table>
<thead>
<tr>
<th>$\Lambda r_b^2$</th>
<th>0.001</th>
<th>0.0025</th>
<th>0.005</th>
<th>0.0075</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_j=\frac{1}{2}$</td>
<td>1.14</td>
<td>2.95</td>
<td>6.11</td>
<td>9.42</td>
<td>12.85</td>
</tr>
<tr>
<td>$\Gamma_j=\frac{3}{2}$</td>
<td>1.38 $\cdot 10^{-5}$</td>
<td>9.07 $\cdot 10^{-5}$</td>
<td>3.84 $\cdot 10^{-4}$</td>
<td>9.03 $\cdot 10^{-4}$</td>
<td>1.67 $\cdot 10^{-3}$</td>
</tr>
<tr>
<td>$\Gamma_j=\frac{5}{2}$</td>
<td>1.2 $\cdot 10^{-10}$</td>
<td>1.94 $\cdot 10^{-9}$</td>
<td>1.56 $\cdot 10^{-8}$</td>
<td>5.9 $\cdot 10^{-8}$</td>
<td>1.2 $\cdot 10^{-7}$</td>
</tr>
</tbody>
</table>

Table: The greybody factors for the first three modes (all the numerical values of $\Gamma_j(\epsilon)$ have been multiplied by a factor of $10^4$).

- For each mode the value of $\Gamma_j$ becomes higher as we increase the value of the cosmological constant $\Lambda$.
- The contribution of the lowest mode $j = 1/2$ to the emission spectra is the dominant one.
- These results are consistent with numerical calculations performed by S. F. Wu et al., Phys. Rev. D 78 (1998) 084010.
**Hawking Radiation**

Result: energy emission rate

*Figure:* The fermion differential energy emission rate for different values of $\Lambda r_b^2$. For the left panel we have set $r_b = 1$, respectively $r_b = 5$ for the right panel. *Obs:* This spectra should be trusted for quantitative results only in the low energy regime.

- The spectrum is enhanced with the increasing value of the cosmological constant;
- The energy emission rate for fermions vanishes in the limit $Energy \to 0$ (as in the case of asymptotically flat BHs);
Conclusions

- Deriving for the first time an analytical formula for low energy greybody factors for fermions emitted by a Schwarzschild-de Sitter black hole;

- For fermions the SdS greybody factors are constant for each mode at very low energies;

- However, for fermions $\Gamma_j$ have a much more complicated dependence on $r_b$ and $r_c$ compared to the scalar case;

- The contribution of the lowest mode $j = 1/2$ to the emission spectra is the dominant one.

- The ratio fermions to scalars emitted by a SdS black hole is approx. $\frac{1}{12}$ (compared to $\frac{1}{8}$ for a Shw. BH).
Thank you for your attention!
to infinity and beyond..