Complete integrability in toric contact geometry

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Outline

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Contact geometry (1)

A $(2n+1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η (called a contact 1-form) on M such that

$$\eta \wedge (d\eta)^{n-1} \neq 0\,.$$

Associated with a contact form η there exists a unique vector field R_η called the *Reeb vector field* defined by the contractions (interior products):

$$\boldsymbol{i}(R_\eta)\eta = 1\,,$$

$$\boldsymbol{i}(R_\eta)d\eta = 0\,.$$

Contact geometry (2)

Every vector field X on M may be decomposed as

$$X = (\boldsymbol{i}(X)\eta)R_\eta + \hat{X}$$

where $\hat{X}$ is the horizontal part of X , i. e. in the kernel of η.

Every 1-form ψ may be decomposed as

$$\psi = (\boldsymbol{i}(R_\eta)\psi)\eta + \hat{\psi}\,,$$

where $\hat{\psi}$ is the semi-basic component of ψ satisfying the relation

$$\boldsymbol{i}(R_\eta)\hat{\psi} = 0\,.$$

Contact geometry (3)

A vector field X on (M, η) is an infinitesimal contact automorphism if and only if there exists a differentiable function ρ such that

$$\mathcal{L}(X)\eta = \rho\eta\,.$$

We shall use the decomposition

$$X_f = fR_\eta + \hat{X}_f\,,$$

where fR_η and $\hat{X}_f$ are, respectively, the vertical and horizontal components with

$$f = \boldsymbol{i}(X_f)\eta\,.$$

Contact geometry (4)

With the help of Cartan's formula connecting the Lie derivative with the interior product,

$$\mathcal{L}(X) = d \circ \boldsymbol{i}(X) + \boldsymbol{i}(X) \circ d\,,$$

in the case of an infinitesimal contact automorphism X_f

$$df + \boldsymbol{i}(X_f)d\eta = \rho\eta\,.$$

Using the properties of the contact form η we have

$$\rho = \boldsymbol{i}(R_\eta)df\,.$$

The condition $\rho = 0$ expresses the fact that f is a *first integral* of the vector field R_η being a constant along the flow of the vector field R_η.

Contact geometry (5)

A chosen contact form η on M defines an isomorphism Φ from the vector space of infinitesimal contact automorphisms onto the set $C^\infty(M)$ of smooth functions on M:

$$\Phi(X_f) = f = \boldsymbol{i}(X_f)\eta\,.$$

Let us remark that the Reeb vector field

$$R_\eta = \Phi^{-1}(1)$$

is an infinitesimal automorphism of the contact form η with $\rho = 0$.

Symplectic Hamiltonian systems (1)

A symplectic manifold is a differential $2n$-dimensional manifold M with a symplectic 2-form Ω closed and non-degenerate. Locally, by Darboux theorem, we can find local coordinates $(q_1, \ldots, q_n; p_1, \ldots, p_n)$ such that

$$\Omega = \sum_{i=1}^{n} = dq_i \wedge dp_i \, .$$

In symplectic geometry a Hamiltonian is a smooth function H such that

$$i(X_H)\Omega = dH \, ,$$

where X_H is an infinitesimal symplectomorphismm i.e.

$$\mathcal{L}(X_H)\Omega = 0 \, .$$

Note: H exists only if the de Rham cohomology class of $i(X_H)\Omega$ vanishes.

Symplectic Hamiltonian systems (2)

A Hamiltonian system is a triple (M, Ω, H) where (M, Ω) is a symplectic manifold and $H \in C^\infty(M, \mathbf{R})$ is a function called the *Hamiltonian function*. A Hamiltonian system is simply a 1-st order differential system associated to the Hamiltonian vector field:

$$\dot{x} = X_H \, .$$

The Poisson bracket of two functions $f, g \in C^\infty(M, \mathbf{R})$ is

$$\{f, g\} := \Omega(X_f, X_g) \, ,$$

where X_f, X_g are the corresponding vector fields to the functions f, g.
If a function f is invariant under the flow of X_H

$$X_H f = \{f, H\} = 0 \, ,$$

it represents a *first integral of motion*.

Symplectic Hamiltonian systems (3)

A Hamiltonian system (M, Ω, H) is *completely integrable* if it possesses n independent integrals of motion $f_1 = H, f_2, \ldots, f_n$ which are pairwise in involution with respect to the Poisson bracket:

$$\{f_i, f_j\} = 0 \quad \text{for all} \quad i, j = 1, \ldots, n\,.$$

According to Arnold-Liouville theorem, for an integrable system with integrals of motion $f_1 = H, f_2, \ldots, f_n$ there exist the coordinates $\vartheta_1, \ldots, \vartheta_n$ known as *angle coordinates* in which the flows of the vector field $X_{f_1}, \ldots, X_{f_n}$ are linear. There are coordinates $I_1, \ldots, I_n$ known as *action coordinates*, complementary to the angle coordinates, such that I_i are integrals of motion.

$(I_1, \ldots, I_n; \vartheta_1, \ldots, \vartheta_n)$ form a Darboux chart and the symplectic form becomes

$$\Omega = \sum_{i=1}^{n} = dI_i \wedge d\vartheta_i\,.$$

Contact Hamiltonian systems (1)

Goal: To give a similar construction in contact geometry.

Note: Unlike the symplectic case, contact structures are automatically Hamiltonian.

In the frame of contact geometry, the vector field $X_f = \Phi^{-1}(f)$ is called the contact Hamiltonian vector field and similarly

$$\dot{x} = X_f \, ,$$

is the *contact Hamiltonian equation* corresponding to f.

X_f is an infinitesimal automorphism of η if and only if df is semi-basic.

Contact Hamiltonian systems (2)

It is often convenient to consider the Reeb vector field R_η as the Hamiltonian vector field with $1 = \eta(R_\eta)$ as the Hamiltonian. In this case the Hamiltonian contact structure is said to be of *Reeb type* and the Hamiltonian is understood to be the constant function 1.

In connection with the isomorphism Φ, the Lie algebra structure of $C^\infty(M)$ is given by the Jacobi bracket

$$\begin{aligned}[f,g]_\eta = \Phi[X_f, X_g] &= -\boldsymbol{i}(X_g)df + f\boldsymbol{i}(R_\eta)dg \\ &= -\boldsymbol{i}(X_f)\boldsymbol{i}(X_g)d\eta + f\boldsymbol{i}(R_\eta)dg - g\boldsymbol{i}(R_\eta)df\,.\end{aligned}$$

Assuming that f and g are first integrals of the vector field R_η we have

$$[f,g]_\eta = d\eta(X_f, X_g)\,.$$

Contact Hamiltonian systems (3)

Also

$$X_{[f,g]_\eta} = [X_f, X_g]_\eta \,,$$
$$X_{[1,g]_\eta} = [R_\eta, X_f]_\eta \,.$$

Notice that Leibniz rule is replaced by

$$[f, gh]_\eta = [f, g]_\eta h + g[f, h]_\eta - [f, 1]_\eta gh \,,$$

which explains the difference between Jacobi brackets and Poisson brackets.

Contact Hamiltonian systems (4)

A Hamiltonian contact structure of Reeb type is said to be *completely integrable* if there exists $(n+1)$ first integrals

$$f_0 = 1, f_1, \ldots, f_n$$

that are independent and in involution.

In addition a completely integrable contact Hamiltonian system is said to be of toric type if the corresponding vector fields

$$X_{f_0} = R_\eta, X_{f_1}, \ldots, X_{f_n}$$

form the Lie algebra of a torus T^{n+1}. The action of a torus T^{n+1} on a contact $(2n+1)$-dimensional manifold (M, η) is completely integrable if it is effective and preserve the contact structure η.

Formulae in local coordinates (1)

Let us consider in a neighborhood U of a point x of M an adapted system of local coordinates $(x^0, x^1, \ldots, x^n, y^1, \ldots, y^n)$. According to Darboux's theorem, in the case of contact geometry, the contact form can be written as

$$\eta = dx^0 - \sum_{k=1}^{n} y^k dx^k \,,$$

and the Reeb vector field defined by η is

$$R_\eta = \frac{\partial}{\partial x^0} \,.$$

In the above adapted system of local coordinates, a vector field can be written as

$$X = a_0 \frac{\partial}{\partial x^0} + \sum_{k=1}^{n} a_k \frac{\partial}{\partial x^k} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial y^k} \,.$$

Formulae in local coordinates (2)

A vector field $X_f = \Phi^{-1}(f)$ has in an local system of coordinates the form

$$X_f = \Big(f - y^k \frac{\partial f}{\partial y^k}\Big)\frac{\partial}{\partial x^0} - \frac{\partial f}{\partial y^k}\frac{\partial}{\partial x^k} + \Big(\frac{\partial f}{\partial x^k} + y^k \frac{\partial f}{\partial x^0}\Big)\frac{\partial}{\partial y^k}\,.$$

Jacobi bracket of two functions f and g may be expressed as

$$\begin{aligned}[f,g]_\eta =& \Big(f - y^k \frac{\partial f}{\partial y^k}\Big)\frac{\partial g}{\partial x^0} - \Big(g - y^k \frac{\partial g}{\partial y^k}\Big)\frac{\partial f}{\partial x^0} \\ &+ \Big(\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial y^k} - \frac{\partial g}{\partial x^k}\frac{\partial f}{\partial y^k}\Big).\end{aligned}$$

Example: Sasaki-Einstein spaces (1)

A contact Riemannian manifold M equipped with a metric g is Sasakian if its metric cone

$$(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2 g)\,,$$

is Kähler. Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. Moreover if the Sasaki manifold is Einstein

$$\mathrm{Ric}_g = 2ng\,,$$

then the Kähler metric cone is Ricci flat $(\mathrm{Ric}_{\bar{g}} = 0)$, i.e. a Calabi-Yau manifold.

Example: Sasaki-Einstein spaces (2)

Sasaki-Einstein space $T^{1,1}$ (1)

The homogeneous toric Sasaki-Einstein 5-dimensional space $T^{1,1}$ is a $U(1)$ bundle over $S^2 \times S^2$. We choose the coordinates (θ_i, ϕ_i) , $i = 1, 2$ to parametrize the two spheres S^2 in the standard way, while the angle $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber. Metric on $T^{1,1}$ may be written as

$$
\begin{aligned}
ds^2(T^{1,1}) =& \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\
&+ \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,.
\end{aligned}
$$

We introduce $\nu = \frac{1}{2}\psi$ so that ν has canonical period 2π. The globally defined contact 1-form η is:

$$
\eta = \frac{1}{3}(2d\nu + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\,.
$$

Example: Sasaki-Einstein spaces (3)

Sasaki-Einstein space $T^{1,1}$ (2)

Reeb vector field R_η has the form

$$R_\eta = \frac{3}{2}\frac{\partial}{\partial \nu} .$$

An effectively acting $\mathbb{T}^3$ action is

$$\mathbf{e}_1 = \frac{\partial}{\partial \phi_1} + \frac{1}{2}\frac{\partial}{\partial \nu} ,$$
$$\mathbf{e}_2 = \frac{\partial}{\partial \phi_2} + \frac{1}{2}\frac{\partial}{\partial \nu} ,$$
$$\mathbf{e}_3 = \frac{\partial}{\partial \nu} ,$$

which preserves the contact structure η.

Example: Sasaki-Einstein spaces (4)

Sasaki-Einstein space $T^{1,1}$ (3)

Let $\mathcal{F} = (f_0, f_1, f_2)$ the set of independent first integrals in involution and $\mathcal{X} = (R_\eta, X_{f_1}, X_{f_2})$ the corresponding set of infinitesimal automorphisms of η. Let T be a compact connected component of the level set $\{f_1 = c_1, f_2 = c_2\}$ and $df_1 \wedge df_2 \neq 0$ on T. Then T is diffeomorphic to a T^3 torus. There exist a neighborhood U of T and a diffeomorphism, $\phi : U \to T^3 \times D$, with $D \in \mathbb{R}^2$,

$$\phi(x) = (\vartheta_0, \vartheta_1, \vartheta_2, y_1, y_2)\,,$$

and the contact form has the canonical expression:

$$\eta_0 = (\phi^{-1})^* \eta = y_0 d\vartheta_0 + y_1 d\vartheta_1 + y_2 d\vartheta_2\,.$$

We refer to the local coordinates (ϑ_i, y_i) as *generalized contact action-angle coordinates*.

Example: Sasaki-Einstein spaces (5)

Sasaki-Einstein space $T^{1,1}$ (4)

$$\eta_0(\frac{\partial}{\partial \vartheta_i}) = y_i$$

are the contact Hamiltonians of the independent set of vector fields $\mathcal{X}$.

It is convenient to choose

$$\vartheta_0 = \frac{2}{3}\nu \,,\, \vartheta_1 = \phi_1 \,,\, \vartheta_2 = \phi_2 \,.$$

First integrals of the Hamiltonian contact structure are

$$f_0 = y_0 \equiv 1 \,,\, f_i = y_i = \frac{1}{3}\cos\theta_i \quad , \quad i = 1, 2 \,,$$

which are independent and in involution

$$[1, f_i]_\eta = [f_i, f_j]_\eta = 0 \quad , \quad i, j = 1, 2 \,.$$

Example: Sasaki-Einstein spaces (6)

Sasaki-Einstein space $T^{1,1}$ (5)

Action of the torus T^3 is given by translations of the angles ϑ_i.

The flows of the set $\mathcal{X}$ on invariant tori is quasi-periodic

$$(\vartheta_0, \vartheta_1, \vartheta_2) \to (\vartheta_0 + t\omega_0, \vartheta_1 + t\omega_1, \vartheta_2 + t\omega_2)\,,$$

where the *frequencies* ω_i depend only on y_i.
In order to construct effectively the flow of X_f and find the frequencies ω_i we define the family of 1-forms

$$\eta_t = \eta_0 + tdf\,,$$

where f is one of the first integrals of the Reeb vector field R_η.
η_t is a contact form also having the Reeb vector field R_η.

Example: Sasaki-Einstein spaces (7)

Sasaki-Einstein space $T^{1,1}$ (6)

Consider the vector field $X = -fR_\eta$ and let ϕ_t the flow of this vector field. Because f is a first integral of the T^3 action, ϕ_t commutes with this action.

Moser's deformation:

$$\mathcal{L}(X)\eta_t = -df = -\frac{\partial \eta_t}{\partial t}\,,$$

which imply

$$\frac{d}{dt}(\phi_t^* \eta_t) = \phi^* \Big(\mathcal{L}(X)\eta_t + \frac{\partial \eta_t}{\partial t} \Big) = 0\,.$$

Therefore $\phi_1^* \eta_1 = \eta_0$ and we can obtain the coordinates in which the 1-form η_t has the canonical expression. Choosing the first integrals $f_i = y_i$, we extract the frequencies:

$$\omega_i = \ln\cos\theta_i \quad , \quad i = 1, 2\,,$$

Example: Sasaki-Einstein spaces (8)

Sasaki-Einstein space $Y^{p,q}$ (1)

Infinite family $Y^{p,q}$ of Einstein-Sasaki metrics on $S^2 \times S^3$ provides supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y^{p,q}$ of an S^1-fibration over $S^2 \times S^2$ with relative prime winding numbers p and q is topologically $S^2 \times S^3$.
Explicit local metric of the 5-dim. $Y^{p,q}$ manifold is given by the line element

$$\begin{aligned} ds^2(Y^{p,q}) = {} & \frac{1-y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{w(y)q(y)}dy^2 \\ & + \frac{q(y)}{9}(d\psi - \cos\theta\, d\phi)^2 \\ & + w(y)\left[d\alpha + \frac{a - 2y + y^2}{6(a-y^2)}[d\psi - \cos\theta\, d\phi]\right]^2, \end{aligned}$$

where a is a constant and

Example: Sasaki-Einstein spaces (9)

Sasaki-Einstein space $Y^{p,q}$ (2)

$$w(y) = \frac{2(a - y^2)}{1 - y} \quad , \quad q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2}$$

For $0 < a < 1$ we can take the range of the angular coordinates (θ, Φ, Ψ) to be $0 \leq \theta \leq \pi\,, 0 \leq \Phi \leq 2\pi\,, 0 \leq \Psi \leq 2\pi$ while y lies between the negative and the smallest positive zeros of $q(y)$.
For any p and q coprime, the space $Y^{p,q}$ is topologically $S^2 \times S^3$ and one may take

$$0 \leq \alpha \leq 2\pi\ell\,,$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}\,.$$

Example: Sasaki-Einstein spaces (10)

Sasaki-Einstein space $Y^{p,q}$ (3)

Sasakian 1-form η is:

$$\eta = -2y d\alpha + \frac{1-y}{3}(d\psi - \cos\theta d\phi)\,.$$

and the Reeb vector field is

$$R_\eta = 3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha}\,.$$

Basis for an effectively acting $\mathbb{T}^3$ action is

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\,,\\
\mathbf{e}_2 &= \frac{\partial}{\partial\phi} - \frac{(p-q)\ell}{2}\frac{\partial}{\partial\alpha}\,,\\
\mathbf{e}_3 &= \ell\frac{\partial}{\partial\alpha}\,.
\end{aligned}$$

Example: Sasaki-Einstein spaces (11)

Sasaki-Einstein space $Y^{p,q}$ (4)

For the canonical forms η and Reeb vector field we introduce the angle variables

$$\vartheta_0 = \frac{\psi}{3}\,,\ \vartheta_1 = -6\alpha - \psi\,,\ \vartheta_2 = \phi\,,$$

and the generalized action variables

$$y_0 \equiv 1\,,\ y_1 = \frac{y}{3}\,,\ y_2 = \frac{y-1}{3}\cos\theta\,.$$

These functions are first integrals of the Hamiltonian contact structure, independent and in involution.

The corresponding set of infinitesimal automorphisms is $\mathcal{X} = (R_\eta, X_{y_1}, X_{y_2})$. The flows of the set $\mathcal{X}$ on invariant tori is quasi-periodic and the evaluation of the frequencies proceeds as in the case of the space $T^{1,1}$.

Outlook

- Contact Hamiltonian dynamics on higher dimensional toric Sasaki-Einstein spaces
- Time-dependent Hamilton function
- Dissipative Hamiltonian systems